Modeling and Analysis of Multiproduct Multistage Manufacturing System for Quality Improvement

Shichang Du, *Member, IEEE*, Rui Xu, and Lin Li, *Member, IEEE*

***Abstract*—The ability to produce multiple types of products using the same manufacturing system plays an essential role in the success of a manufacturing enterprise. Meanwhile, most complex manufacturing systems include many stages, called multistage manufacturing systems (MMSs). MMS has a cascade property, which means the product quality is not only affected by the current stage, but is also related to the outputs of the upstream stage. Multiproduct MMSs (M^3Ss) have been widely applied in industry. Thus, this paper is devoted to modeling and analyzing steady-state M^3Ss for quality improvement. The discrete Markov model for single-product-multistage system is extended to novel Markov models for multiproduct-two-stage systems and multiproduct-multistage systems by calculating state transition probabilities of six key manufacturing parameters to obtain an acceptable product quality probability. Based on the developed models, product sequence analysis is carried out to obtain the best product sequence under Bernoulli conditions and Bernoulli relaxation conditions, and bottleneck analysis under Bernoulli conditions is conducted to identify the machine and/or parameters whose improvement can lead to the largest quality improvement. The applicability of the proposed models is validated through numerical experiments and a case study using real-world data.**

***Index Terms*—Bottleneck, Markov model, multiple products, multistage manufacturing system (MMS), quality control.**

NOMENCLATURE

M_i	ith stage in multistage manufacturing systems (MMSs).
M'_i	Stage merged by the first i stages in MMSs.
k_j	Batch size of product j.
K	Total amount of products produced in one batch.
g_i	M_i or M'_i is producing a good product.
d_i	M_i or M'_i is producing a defective product.
S^l	One possible sequence when producing multiple products.
S^l_m	mth type of product in sequence S^l.
$S^l_{m,j}$	jth product internal of S^l_m.
$g_{i,s^l_m,j}$	M_i or M'_i is in good state g_i when producing product $S^l_{m,j}$.
$d_{i,s^l_m,j}$	M_i or M'_i is in defective state d_i when producing product $S^l_{m,j}$.
α_1	Probability for M_1 to transit from state g_1 to state d_1.
α_{1,S^l_p,S^l_r}	Probability for M_1 to transit from state g_1 to state d_1 when switching from producing S^l_p to producing S^l_r.
β_1	Probability for M_1 to transit from state d_1 to state g_1.
β_{1,S^l_p,S^l_r}	Probability for M_1 to transit from state d_1 to state g_1 when switching from producing S^l_p to producing S^l_r.
α'_i	Probability for M_i to transit from state g_i to state $d_i (i \geq 2)$.
α'_{i,S^l_p,S^l_r}	Probability for M'_i to transit from state g_i to state $d_i (i \geq 2)$ when switching from producing S^l_p to producing S^l_r.
β'_i	Probability for M'_i to transit from state g_i to state $g_i (i \geq 2)$.
β'_{i,S^l_p,S^l_r}	Probability for M'_i to transit from state g_i to state $g_i (i \geq 2)$ when switching from producing S^l_p to producing S^l_r.
γ_i	In case of good incoming parts, the probability for M_i to transit from state g_i to state d_i.
γ_{i,S^l_p,S^l_r}	In case of good incoming parts, the probability for M_i to transit from state g_i to state d_i when switching from producing S^l_p to producing S^l_r.
μ_i	In case of good incoming parts, the probability for M_i to transit from state d_i to state g_i.
μ_{i,S^l_p,S^l_r}	In case of good incoming parts, the probability for M_i to transit from state d_i to state g_i when switching from producing S^l_p to producing S^l_r.
η_i	In case of defective incoming parts, the probability for M_i to transit from state g_i to state d_i.
η_{i,S^l_p,S^l_r}	In case of defective incoming parts, the probability for M_i to transit from state g_i to state d_i when switching from producing S^l_p to producing S^l_r.
θ_i	In case of defective incoming parts, the probability for M_i to transit from state d_i to state g_i.

Manuscript received July 23, 2016; accepted September 20, 2016. Date of publication October 19, 2016; date of current version April 13, 2018. This work was supported in part by the Major Program of the National Natural Science Foundation of China under Grant 51535007 and in part by the National Natural Science Foundation of China under Grant 51275558. This paper was recommended by Associate Editor A. Trappey. *(Corresponding author: Shichang Du.)*

S. Du and R. Xu are with the Department of Industrial Engineering and Management, Shanghai Jiao Tong University, Shanghai 200240, China (e-mail: lovbin@sjtu.edu.cn; lucyruixu@126.com).

L. Li is with the Department of Industrial and Manufacturing Engineering, University of Illinois at Chicago, Chicago, IL 60607 USA (e-mail: linli@uic.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSMC.2016.2614766

θ_{i,S_p^l,S_r^l} In case of defective incoming parts, the probability for M_i to transit from state d_i to state g_i when switching from producing S_p^l to producing S_r^l.

A_i Matrix of state transition probabilities for the system with i stages.

$A_{m,j}$ Matrix of state transition probabilities for the system with i stages when producing the jth product internal of S_m^l.

X_i Matrix of steady-state probabilities for the system with i stages.

$X_{m,j}$ Matrix of steady-state probabilities for the system with i stages when producing the jth product internal of S_m^l.

P Probability of the system in one certain steady state.

$P(g_i)$ Probability of the system with i stages in a good state.

$P(g_{\text{bp}}^l)$ Probability that the system produces good products under batch production (BP) with sequence S^l.

$P(g_{\text{sp}}^l)$ Probability that the system produces good products under single production (SP) with sequence S^l.

S_{γ_i} Sensitivity of $P(g_i)$ with respect to γ_i.

S_{μ_i} Sensitivity of $P(g_i)$ with respect to μ_i.

S_{η_i} Sensitivity of $P(g_i)$ with respect to η_i.

S_{θ_i} Sensitivity of $P(g_i)$ with respect to θ_i.

S_{α_1} Sensitivity of $P(g_i)$ with respect to α_1.

S_{β_1} Sensitivity of $P(g_i)$ with respect to β_1.

δ_1 Maximum difference between 1 and $(\alpha_{1,s_i^l,s_j^l} + \beta_{1,s_i^l,s_j^l})$.

δ_2 Maximum difference between 1 and $(\gamma_{2,s_i^l,s_j^l} + \mu_{2,s_i^l,s_j^l})$.

δ_3 Maximum difference between 1 and $(\eta_{2,s_i^l,s_j^l} + \theta_{2,s_i^l,s_j^l})$.

I. Introduction

THE ABILITY to produce multiple types of products using the same manufacturing system plays an essential role in the success of a manufacturing enterprise. Product flexibility increases the rapid responsiveness of a system; it takes full advantage of available system resources to produce multiple types of products using the same manufacturing system that deals with internal and external production uncertainties with time. Meanwhile, most complex manufacturing systems involve many stages, called MMSs. As products move through these stages, the variations in product quality are usually introduced and propagated. Multiproduct MMSs (M^3Ss) are becoming more and more popular and necessary and have been widely applied in automobile vehicles, engine, aerospace, electronics, and appliance industry.

In M^3Ss, the final product quality is dependent on not only product flexibility but also product quality propagation. For example, flexible fixtures play an important role to enable flexibility of the whole manufacturing system, and they are programmable in order to hold and clamp different types of products in a same manufacturing system. The fixtures locating accuracy determine the product quality. When the product is changed from one type to another one, the fixtures need to adapt themselves to the desired corresponding positions. Since there exist relocation errors of the fixtures, their conditions could be changed from "good" to "bad." Assume there are products A and B machined in two-stage system (see Fig. 1). If the fixture is in good condition for product A and the subsequent product is also product A at stage 1, then a good quality product would be produced. However, if the subsequent product is changed from products A to B, then the fixture needs to readjust its position (there exists a transition probability from good position to bad position) and either good quality or defective products at stage 1 may be produced. The defective product B may be produced at stage 1. Thus, the product quality at stage 1 is dependent on not only the quality of the previous product but also the product type, namely, the product flexibility can affect the quality. Meanwhile, if the characteristic of defective product B (caused by product flexibility) produced at stage 1 is used as the locating datum to machine other characteristics of product B at stage 2, then the quality variation of product B would propagate from stages 1 to 2. Flexibility can affect the quality propagation. Therefore, both product flexibility and quality propagation could significantly affect the final product quality in M^3Ss. However, there has been no analytical model to investigate and improve product quality considering both product flexibility and product quality propagation in M^3Ss.

Fig. 1. Two-product two-stage manufacturing system.

Modeling and analysis of MMSs for quality propagation has received intensive investigation. One of the most popular analytical models used for quality improvement is state space model [1]. After that, a great deal of extended research on quality propagation based on state space model has been conducted. Detailed descriptions of existing research on the state space model are provided in a monograph [2] and a survey [3]. In another research line of analytical methods, a Markov model has been widely used as an analytical tool to investigate the interactions between the manufacturing system and product quality [4]. The related literatures are reviewed in [5]. However, in spite of above efforts, the current Markov models are explored only for single-product-multistage systems without product flexibility [6], [7] or multiproduct-single-stage system without quality propagation [8]–[11]. To the best knowledge of the authors, there lacks an efficient method to model and analyze M^3Ss which features both quality propagation and product flexibility.

The main contribution of this paper is to propose novel Markov models for steady-state M^3Ss for quality improvement, which take both product flexibility and product quality propagation into consideration.

The remainder of this paper is organized as follows. Section II reviews the related literature. Section III introduces

M^{3}Ss and formulates the problem. Two Markov models for multiproduct-two-stage system and multiproduct-multistage system are developed, respectively, in Section IV. Product sequence and quality improvement analysis are carried out in Sections V and VI. A case study is presented to illustrate the proposed models and their applicability in Section VII. Finally, the conclusion is given in Section VIII.

II. Literature Review

Product flexibility has been widely recognized as a critical component to achieve a competitive advantage in the market place. Flexible manufacturing systems (FMSs) have been widely designed and applied [12]–[14]. Detailed descriptions of existing research on manufacturing flexibility analysis methods and applications were reviewed in [15]–[18].

Modeling and analysis of product quality propagation in MMSs have attracted substantial research attention in recent decades. Data-driven methods focus on investigating patterns from massive historical quality datasets to model the relationships between product quality and manufacturing systems [19]–[22].

Unlike data-driven models, analytical models employ offline analysis of MMSs based on fundamental physical laws. The state space model is one of the most popular analytical models used for quality propagation analysis [1]. It is further investigated in 3-D assembly systems [23]–[26] and machining systems [27]–[33]. Detailed descriptions of the existing research on state space model are provided in a monograph [2] and a survey [3]. However, analysis of complex MMSs using the state space model based on physical laws is often intractable [3], and such analysis either relies on complicated kinematics models of manufacturing systems, or is only applicable to deal with dimensional errors and the application area is limited [34], [35].

In another research concentration, Markov models have been widely used as analytical tools to investigate the product quality. The related literature and empirical evidences show that manufacturing system design has a significant impact on product quality [5]. Modeling and analysis of manufacturing systems using Markov models for product quality improvement have received more and more research attention.

Some Markov models are developed to analyze product quality propagation in single-product-multistage systems. Markov-chain-based quality propagation models are developed to evaluate quality propagation in automotive paint shops [6] and battery assembly process [7]. The analytical methods using Markov models are proposed to evaluate three cases of long manufacturing lines [36]. The impact of manufacturing system design on product quality is investigated through a case study at an automotive paint shop [37]. Also, analytical methods are proposed for the joint design of quality and manufacturing parameters [38] and investigate joint production and quality control in manufacturing systems with random demand [39].

Some other Markov models are derived to analyze the quality of multiproduct-single-stage systems. The early Markov model is developed to study how production system design, quality and productivity are inter-related [40]. Then a Markov model to evaluate quality performance of multiproduct-single-stage systems is presented in [41]. Based on Markov chain processes, some analytical methods are developed to evaluate quality performance in multiproduct manufacturing systems with BP in [8].

Several research studies on product sequencing and bottlenecks using Markov models in multiproduct-single-stage systems to improve product quality have been conducted. The impact of product sequencing and batch policies on product quality is investigated and some insights to achieve better quality are presented [9]. Quality bottleneck analysis is carried out based on data from a factory floor [10], [11]. An arrow-based bottleneck identification method is presented in [6]. A Markov model is applied to characterize a furniture assembly system in [42]. A Markov model is explored to control dynamic energy for energy efficiency improvement of sustainable manufacturing systems [43].

In spite of the above efforts, M^{3}Ss with quality propagation and product flexibility still lacks an in-depth study, and there is no efficient Markov model to analyze these systems for quality improvement. The goal of this paper is to contribute to this end.

III. Model Assumptions and Problem Formulation

Consider an n-product-r-stage system; the following assumptions define the processing stages, product types, and their interactions in the Markov models.

1) The manufacturing system consists of r stages which can produce n different types of products. The corresponding batch size for product j is $k_j (1 \leq j \leq n)$. The total amount of products produced in one batch is $K = \sum_{i=1}^{n} k_i$.
2) We only study the working or production period of the system. Machine breakdowns are not considered.
3) Define the stage $M_i (i = 1, 2, \ldots, r)$ in a good state g_i or a defective state d_i if it is producing a product with good quality or defective quality at time t. The quality of incoming product is characterized by good state g_i or defective state d_i with probabilities $P(g_i)$ and $P(d_i)$, respectively.
4) There exist quality degradation and quality correction in the system. The quality might get worse or better after a certain stage. The quality of the incoming parts for $M_i (i \geq 2)$ at time $(t + 1)$ depends on the state of M_{i-1} at time t. The states g_{i-1} and d_{i-1} for M_{i-1} at time t means good and defective parts coming for M_i at time $(t + 1)$, respectively.
5) When M_1 is in good state g_1, it has probability α_1 to transit to defective state d_1 and probability $(1 - \alpha_1)$ to good state g_1. When M_1 is in defective state d_1, it has probability β_1 to transit to good state g_1 and probability $(1 - \beta_1)$ to defective state d_1.
6) With good incoming parts, when $M_i (i \geq 2)$ is in good state g_i, it has probability γ_i to transit to defective state d_i and probability $(1 - \gamma_i)$ to good state g_i. When M_i is in

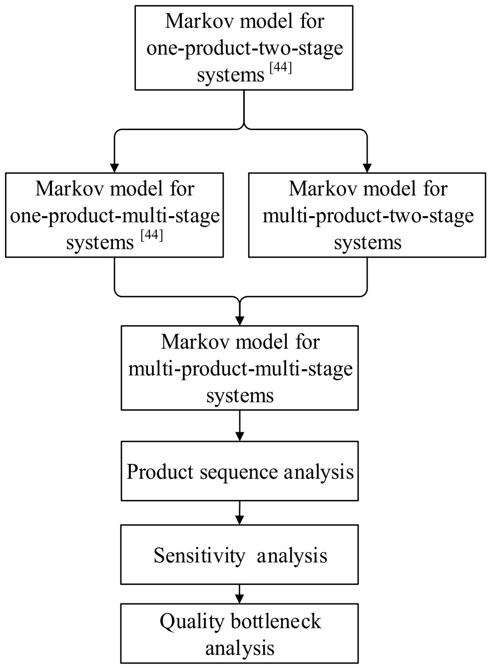

Fig. 2. Framework of the proposed method.

defective state d_i, it has probability μ_i to transit to good state g_i and probability $(1-\mu_i)$ to defective state d_i. With defective incoming parts, when $M_i (i \geq 2)$ is in good state g_i, it has probability η_i to transit to defective state d_i and probability $(1-\eta_i)$ to good state g_i. When M_i is in defective state d_i, it has probability θ_i to transit to good state g_i and probability $(1-\theta_i)$ to defective state d_i.

7) For the system producing n different types of products, there are $(n-1)!$ possible production sequences. For a certain sequence $S^l = \{S_1^l, S_2^l, \ldots, S_n^l\} (l = 1, 2, \ldots, (n-1)!)$, S_m^l denotes the mth type of product in this sequence, where $m \in \{1, 2, \ldots, n\}$.

8) For a certain product S_m^l in sequence S^l, the system will not transit to processing another type of product $S_n^l (n \neq m)$ until it finishes processing the last one $k_{S_m^l}$ of product S_m^l.

9) When M_1 or $M_i (i = 2, 3, \ldots, r)$ is processing $S_{m,j}^l$ at time t, it may maintain processing the same type of product $S_{m,j+1}^l$ or switch to processing another type of product $S_{m+1,1}^l$ at $(t+1)$. Then the corresponding transition probabilities for M_1 and M_i are denoted as α_{1,S_p^l,S_r^l}, β_{1,S_p^l,S_r^l} and γ_{2,S_p^l,S_r^l}, μ_{2,S_p^l,S_r^l}, η_{2,S_p^l,S_r^l}, and θ_{2,S_p^l,S_r^l} $(p, r = 1, 2, \ldots, n)$. When $p = r$, these probabilities imply the internal transition probabilities of the same type of product. When $p \neq r$, they indicate the external transition probabilities between different types of products.

We refer to $\alpha_1, \gamma_i, \eta_i (i \geq 2)$ as quality failure probabilities and $(i \geq 2)\beta_1, \mu_i, \theta_i (i \geq 2)$ as quality repair probabilities. Similar to throughput analysis and in accordance with some quality analysis work based on Markov model [6], [7], [41], we assume that all these transition probabilities are constant. Actually in real manufacturing systems, machines have stable production periods during which the state transitions can be seen as stable.

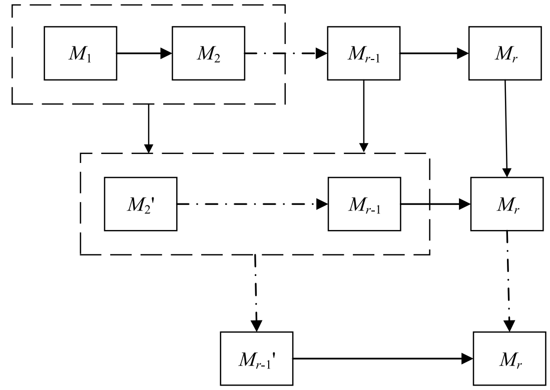

Fig. 3. Iterative procedure for multistage systems.

The problem addressed is then formulated as follows.

Problem: Under the above assumptions 1)–9), develop a proper method to evaluate the steady-state quality performance of M^{3}Ss as a function of system parameters, investigate the sequence properties, perform sensitivity analysis and identify quality bottlenecks.

The solutions to the problem are given in Sections IV–VI. The framework of the proposed method is illustrated in Fig. 2. Based on the Markov model for single-product-two-stage systems in [44], a Markov model for multiproduct-two-stage systems is developed. Then this model is extended to multiproduct-multistage systems. Product sequence analysis and quality improvement analysis are conducted based on the developed models.

IV. Markov Model for M^{3}Ss

A. Review of the Markov Model for Single-Product-Multistage-Systems

Based on the assumptions 1)–9) and the work of [44], for a two-stage system producing a given type of product $S_{m,j}^l$, the transition probabilities and the matrix of state transition probabilities are presented (1), as shown at the top of the next page.

The matrix of steady state probabilities is denoted as

$$X_2 = [\,P(g_1g_2) \quad P(d_1g_2) \quad P(g_1d_2) \quad P(d_1d_2)\,]. \tag{2}$$

Based on the Markov model, we have

$$X_2A_2 = X_2 \tag{3}$$

$$P(g_1g_2) + P(d_1g_2) + P(g_1d_2) + P(d_1d_2) = 1 \tag{4}$$

and

$$P(g_2) = P(g_1g_2) + P(d_1g_2). \tag{5}$$

Equation (5) represents the probability that the system is producing a good product. This probability can be seen as an indicator to evaluate the quality performance of the system.

The model for single-product-two-stage systems can be extended to single-product-r-stage systems $(r \geq 3)$ by applying the iteration method in [44] which has been validated

$$A_2 = \begin{bmatrix} \left(1-\alpha_{1,S^l_{m,j},S^l_{m,j}}\right)\left(1-\gamma_{2,S^l_{m,j},S^l_{m,j}}\right) & \alpha_{1,S^l_{m,j},S^l_{m,j}}\left(1-\gamma_{2,S^l_{m,j},S^l_{m,j}}\right) & \left(1-\alpha_{1,S^l_{m,j},S^l_{m,j}}\right)\gamma_{2,S^l_{m,j},S^l_{m,j}} & \alpha_{1,S^l_{m,j},S^l_{m,j}}\gamma_{2,S^l_{m,j},S^l_{m,j}} \\ \beta_{1,S^l_{m,j},S^l_{m,j}}\left(1-\eta_{2,S^l_{m,j},S^l_{m,j}}\right) & \left(1-\beta_{1,S^l_{m,j},S^l_{m,j}}\right)\left(1-\eta_{2,S^l_{m,j},S^l_{m,j}}\right) & \beta_{1,S^l_{m,j},S^l_{m,j}}\eta_{2,S^l_{m,j},S^l_{m,j}} & \left(1-\beta_{1,S^l_{m,j},S^l_{m,j}}\right)\eta_{2,S^l_{m,j},S^l_{m,j}} \\ \left(1-\alpha_{1,S^l_{m,j},S^l_{m,j}}\right)\mu_{2,S^l_{m,j},S^l_{m,j}} & \alpha_{1,S^l_{m,j}S^l_{m,j}}\mu_{2,S^l_{m,j}S^l_{m,j}} & \left(1-\alpha_{1,S^l_{m,j},S^l_{m,j}}\right)\left(1-\mu_{2,S^l_{m,j},S^l_{m,j}}\right) & \left(1-\alpha_{1,S^l_{m,j},S^l_{m,j}}\right)\mu_{2,S^l_{m,j},S^l_{m,j}} \\ \beta_{1,S^l_{m,j},S^l_{m,j}}\theta_{2,S^l_{m,j},S^l_{m,j}} & \left(1-\beta_{1,S^l_{m,j},S^l_{m,j}}\right)\theta_{2,S^l_{m,j},S^l_{m,j}} & \beta_{1,S^l_{m,j},S^l_{m,j}}\left(1-\theta_{2,S^l_{m,j},S^l_{m,j}}\right) & \left(1-\beta_{1,S^l_{m,j},S^l_{m,j}}\right)\left(1-\theta_{2,S^l_{m,j},S^l_{m,j}}\right) \end{bmatrix} \tag{1}$$

by extensive numerical experiments. The general iterative procedure is illustrated in Fig. 3 and the main steps are presented as follows.

1) Merge M_1 and M_2 to one stage M_2', and gain the quality of the new two-stage system $M_2' - M_3$ based on the model for one-product-two-stage system.
2) Merge M_2' and M_3 to one stage M_3', and gain the quality of the new two-stage system $M_3' - M_4$. Continue this iterative process until the first $(r-1)$ stages are merged to one stage M_{r-1}' and gain the quality of the final two-stage system $M_{r-1}' - M_r$.

During the iterative process of single-product-r-stage systems, any two-stage system $M_i' - M_{i+1}$ has six basic parameters. They are $\gamma_{i+1,S^l_{m,j},S^l_{m,j}}$, $\mu_{i+1,S^l_{m,j},S^l_{m,j}}$, $\eta_{i+1,S^l_{m,j},S^l_{m,j}}$, and $\theta_{i+1,S^l_{m,j},S^l_{m,j}}$ describing the characteristics of M_{i+1} and $\alpha'_{i,S^l_{m,j},S^l_{m,j}}$, $\beta'_{i,S^l_{m,j},S^l_{m,j}}$ describing the characteristics of the merged machine M_i'. $\alpha'_{i,S^l_{m,j},S^l_{m,j}}$, $\beta'_{i,S^l_{m,j},S^l_{m,j}}$ can be calculated by

$$\alpha'_{i,S^l_{m,j},S^l_{m,j}} = \frac{P(g_{i-1}g_i)\gamma_{i,S^l_{m,j},S^l_{m,j}} + P(d_{i-1}g_i)\eta_{i,S^l_{m,j},S^l_{m,j}}}{P(g_{i-1}g_i) + P(d_{i-1}g_i)} \tag{6}$$

$$\beta'_{i,S^l_{m,j},S^l_{m,j}} = \frac{P(g_{i-1}d_i)\mu_{i,S^l_{m,j}S^l_{m,j}} + P(d_{i-1}d_i)\theta_{i,S^l_{m,j},S^l_{m,j}}}{P(g_{i-1}d_i) + P(d_{i-1}d_i)}. \tag{7}$$

The matrix of state transition probabilities (8), as shown at the top of the next page.

The corresponding matrix of the steady-state probabilities is

$$X_{i+1} = [\,P(g_i g_{i+1}) \quad P(d_i g_{i+1}) \quad P(g_i d_{i+1}) \quad P(d_i d_{i+1})\,]. \tag{9}$$

According to the definition of a Markov chain, we obtain

$$X_{i+1}A_{i+1} = X_{i+1} \tag{10}$$

$$P(g_i g_{i+1}) + P(d_i g_{i+1}) + P(g_i d_{i+1}) + P(d_i d_{i+1}) = 1. \tag{11}$$

The final probability of producing a product with good quality for the merged two-stage system is

$$P(g_{i+1}) = P(g_i g_{i+1}) + P(d_i g_{i+1}). \tag{12}$$

B. Proposed Markov Model for Multiproduct-Two-Stage Systems

Based on the Markov model for single-product-two-stage systems, by further considering product flexibility, we can extend the model to multiproduct-two-stage systems. It can be seen from assumption 9) that different positions of a part in the batch may lead to different transition probabilities.

1) When M_2 is processing the first part of a certain product sequence $S^l_{m,1} (2 \le m \le n)$, the corresponding probabilities are α_{1,S^l_m,S^l_m}, β_{1,S^l_m,S^l_m}, $\gamma_{2,S^l_{m-1},S^l_m}$, μ_{2,S^l_{m-1},S^l_m}, η_{2,S^l_{m-1},S^l_m}, and $\theta_{2,S^l_{m-1},S^l_m}$. The matrix of steady state probability $X_{m,1}$ and transition probabilities $A_{m,1}$ are (13) and (14), as shown at the top of the next page. When $m = 1$, i.e., M_2 is processing the first part of the whole batch cycle, the corresponding probabilities are α_{1,S^l_1,S^l_1}, β_{1,S^l_1,S^l_1}, γ_{2,S^l_n,S^l_1}, μ_{2,S^l_n,S^l_1}, η_{2,S^l_n,S^l_1}, θ_{2,S^l_n,S^l_1}, and (15) and (16), as shown at the top of the next page.
2) When M_2 is processing the jth part of a certain product $S^l_{m,j}$, $m = 1, 2, \ldots, n$, $j = 2, 3, \ldots, k_{S^l_i} - 1$, the corresponding probabilities are α_{1,S^l_m,S^l_m}, β_{1,S^l_m,S^l_m}, γ_{2,S^l_m,S^l_m}, μ_{2,S^l_m,S^l_m}, η_{2,S^l_m,S^l_m}, θ_{2,S^l_m,S^l_m}, and (17) and (18), as shown at the top of the next page.
3) When M_2 is processing the last part of a certain product sequence $S^l_{m,k_{S^l_m}}$ $(m = 1, 2, \ldots, n)$, the corresponding probabilities are $\alpha_{1,S^l_m,S^l_{m+1}}$, $\beta_{1,S^l_m,S^l_{m+1}}$, γ_{2,S^l_m,S^l_m}, μ_{2,S^l_m,S^l_m}, η_{2,S^l_m,S^l_m}, θ_{2,S^l_m,S^l_m}, and (19) and (20), as shown at the top of the next page. When $m = n$, it means one batch cycle is finished. Then $(m+1)$ would be 1 as a new batch cycle starts.

Based on the above analysis, we have the matrix of steady state probabilities and matrix of state transition probabilities in the form of a block matrix

$$X = \Big[\mathrm{X}_{1,1}, X_{1,2}, \ldots, X_{m,k_{S^l_m}}, X_{m+1,1}, \ldots, X_{n-1,k_{S^l_{n-1}}}, \\ X_{n,1}, \ldots, X_{n,k_{S^l_n}-1}, X_{n,k_{S^l_n}}\Big]^T \tag{21}$$

$$A = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & A_{1,1} \\ A_{1,2} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & A_{m,k_{S^l_m}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & A_{m+1,1} & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & A_{m+1,2} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & A_{n,k_{S^l_n}} & 0 \end{bmatrix}. \tag{22}$$

The sum of all the steady state probabilities equals 1

$$\sum_{m=1}^{n}\sum_{j=1}^{k_{s^l_m}} P\left(g_{2,s^l_m,j}\right) + \sum_{m=1}^{n}\sum_{j=1}^{k_{s^l_m}} P\left(d_{2,s^l_m,j}\right) = 1 \tag{23}$$

and

$$AX = X. \tag{24}$$

$$A_{i+1} = \begin{bmatrix} \left(1-\alpha'_{1,S^l_{m,j},S^l_{m,j}}\right)\left(1-\gamma_{2,S^l_{m,j},S^l_{m,j}}\right) & \alpha'_{1,S^l_{m,j},S^l_{m,j}}\left(1-\gamma_{2,S^l_{m,j},S^l_{m,j}}\right) & \left(1-\alpha_{1,S^l_{m,j},S^l_{m,j}}\right)\gamma_{2,S^l_{m,j},S^l_{m,j}} & \alpha'_{1,S^l_{m,j},S^l_{m,j}}\gamma_{2,S^l_{m,j},S^l_{m,j}} \\ \beta'_{1,S^l_{m,j},S^l_{m,j}}\left(1-\eta_{2,S^l_{m,j},S^l_{m,j}}\right) & \left(1-\beta'_{1,S^l_{m,j},S^l_{m,j}}\right)\left(1-\eta_{2,S^l_{m,j},S^l_{m,j}}\right) & \beta'_{1,S^l_{m,j},S^l_{m,j}}\eta_{2,S^l_{m,j},S^l_{m,j}} & \left(1-\beta'_{1,S^l_{m,j},S^l_{m,j}}\right)\eta_{2,S^l_{m,j},S^l_{m,j}} \\ \left(1-\alpha'_{1,S^l_{m,j},S^l_{m,j}}\right)\mu_{2,S^l_{m,j},S^l_{m,j}} & \alpha'_{1,S^l_{m,j}S^l_{m,j}}\mu_{2,S^l_{m,j}S^l_{m,j}} & \left(1-\alpha'_{1,S^l_{m,j},S^l_{m,j}}\right)\left(1-\mu_{2,S^l_{m,j},S^l_{m,j}}\right) & \left(1-\alpha'_{1,S^l_{m,j},S^l_{m,j}}\right)\mu_{2,S^l_{m,j},S^l_{m,j}} \\ \beta'_{1,S^l_{m,j},S^l_{m,j}}\theta_{2,S^l_{m,j},S^l_{m,j}} & \left(1-\beta'_{1,S^l_{m,j},S^l_{m,j}}\right)\theta_{2,S^l_{m,j},S^l_{m,j}} & \beta'_{1,S^l_{m,j},S^l_{m,j}}\left(1-\theta_{2,S^l_{m,j},S^l_{m,j}}\right) & \left(1-\beta'_{1,S^l_{m,j},S^l_{m,j}}\right)\left(1-\theta_{2,S^l_{m,j},S^l_{m,j}}\right) \end{bmatrix} \tag{8}$$

$$X_{m,1} = \left[P\left(g_{1,S^l_{m,2}}g_{2,S^l_{m,1}}\right) \quad P\left(d_{1,S^l_{m,2}}g_{2,S^l_{m,1}}\right) \quad P\left(g_{1,S^l_{m,2}}d_{2,S^l_{m,1}}\right) \quad P\left(g_{1,S^l_{m,2}}d_{2,S^l_{m,1}}\right) \right] \tag{13}$$

$$A_{i,1} = \begin{bmatrix} \left(1-\alpha_{1,S^l_m,S^l_m}\right)\left(1-\gamma_{2,S^l_{m-1},S^l_m}\right) & \beta_{1,S^l_m,S^l_m}\left(1-\eta_{2,S^l_{m-1},S^l_m}\right) & \left(1-\alpha_{1,S^l_m,S^l_m}\right)\mu_{2,S^l_{m-1},S^l_m} & \beta_{1,S^l_m,S^l_m}\theta_{2,S^l_{m-1},S^l_m} \\ \alpha_{1,S^l_m,S^l_m}\left(1-\gamma_{2,S^l_{m-1},S^l_m}\right) & \left(1-\beta_{1,S^l_m,S^l_m}\right)\left(1-\eta_{2,S^l_{m-1},S^l_m}\right) & \alpha_{1,S^l_m,S^l_m}\mu_{2,S^l_{m-1},S^l_m} & \left(1-\beta_{1,S^l_m,S^l_m}\right)\theta_{2,S^l_{m-1},S^l_m} \\ \left(1-\alpha_{1,S^l_m,S^l_m}\right)\gamma_{2,S^l_{m-1},S^l_m} & \beta_{1,S^l_m,S^l_m}\eta_{2,S^l_{m-1},S^l_m} & \left(1-\alpha_{1,S^l_m,S^l_m}\right)\left(1-\mu_{2,S^l_{m-1},S^l_m}\right) & \beta_{1,S^l_m,S^l_m}1-\theta_{2,S^l_{m-1},S^l_m}) \\ \alpha_{1,S^l_m,S^l_m}\gamma_{2,S^l_{m-1},S^l_m} & \left(1-\beta_{1,S^l_m,S^l_m}\right)\eta_{2,S^l_{m-1},S^l_m} & \alpha_{1,S^l_m,S^l_m}\left(1-\mu_{2,S^l_{m-1},S^l_m}\right) & \left(1-\beta_{1,S^l_m,S^l_m}\right)\left(1-\theta_{2,S^l_{m-1},S^l_m}\right) \end{bmatrix} \tag{14}$$

$$X_{1,1} = \left[P\left(g_{1,S^l_{1,2}}g_{2,S^l_{1,1}}\right) \quad P\left(d_{1,S^l_{1,2}}g_{2,S^l_{1,1}}\right) \quad P\left(g_{1,S^l_{1,2}}d_{2,S^l_{1,1}}\right) \quad P\left(g_{1,S^l_{1,2}}d_{2,S^l_{1,1}}\right) \right] \tag{15}$$

$$A_{1,1} = \begin{bmatrix} \left(1-\alpha_{1,S^l_1,S^l_1}\right)\left(1-\gamma_{2,S^l_n,S^l_1}\right) & \beta_{1,S^l_1,S^l_1}\left(1-\eta_{2,S^l_n,S^l_1}\right) & \left(1-\alpha_{1,S^l_1,S^l_1}\right)\mu_{2,S^l_n,S^l_1} & \beta_{1,S^l_1,S^l_1}\theta_{2,S^l_n,S^l_1} \\ \alpha_{1,S^l_1,S^l_1}\left(1-\gamma_{2,S^l_n,S^l_1}\right) & \left(1-\beta_{1,S^l_1,S^l_1}\right)\left(1-\eta_{2,S^l_n,S^l_1}\right) & \alpha_{1,S^l_1,S^l_1}\mu_{2,S^l_n,S^l_1} & \left(1-\beta_{1,S^l_1,S^l_1}\right)\theta_{2,S^l_n,S^l_1} \\ \left(1-\alpha_{1,S^l_1,S^l_1}\right)\gamma_{2,S^l_n,S^l_1} & \beta_{1,S^l_1,S^l_1}\eta_{2,S^l_n,S^l_1} & \left(1-\alpha_{1,S^l_1,S^l_1}\right)\left(1-\mu_{2,S^l_n,S^l_1}\right) & \beta_{1,S^l_1,S^l_1}\left(1-\theta_{2,S^l_n,S^l_1}\right) \\ \alpha_{1,S^l_1,S^l_1}\gamma_{2,S^l_n,S^l_1} & \left(1-\beta_{1,S^l_1,S^l_1}\right)\eta_{2,S^l_n,S^l_1} & \alpha_{1,S^l_1,S^l_1}\left(1-\mu_{2,S^l_n,S^l_1}\right) & \left(1-\beta_{1,S^l_1,S^l_1}\right)\left(1-\theta_{2,S^l_n,S^l_1}\right) \end{bmatrix} \tag{16}$$

$$X_{m,j} = \left[P\left(g_{1,S^l_{m,j+1}}g_{2,S^l_{m,j}}\right) \quad P\left(d_{1,S^l_{m,j+1}}g_{2,S^l_{m,j}}\right) \quad P\left(g_{1,S^l_{m,j+1}}d_{2,S^l_{m,j}}\right) \quad P\left(g_{1,S^l_{m,j+1}}d_{2,S^l_{m,j}}\right) \right] \tag{17}$$

$$A_{m,j} = \begin{bmatrix} \left(1-\alpha_{1,S^l_m,S^l_m}\right)\left(1-\gamma_{2,S^l_m,S^l_m}\right) & \beta_{1,S^l_m,S^l_m}\left(1-\eta_{2,S^l_m,S^l_m}\right) & \left(1-\alpha_{1,S^l_m,S^l_m}\right)\mu_{2,S^l_m,S^l_m} & \beta_{1,S^l_m,S^l_m}\theta_{2,S^l_m,S^l_m} \\ \alpha_{1,S^l_m,S^l_m}\left(1-\gamma_{2,S^l_m,S^l_m}\right) & \left(1-\beta_{1,S^l_m,S^l_m}\right)\left(1-\eta_{2,S^l_m,S^l_m}\right) & \alpha_{1,S^l_m,S^l_m}\mu_{2,S^l_m,S^l_m} & \left(1-\beta_{1,S^l_m,S^l_m}\right)\theta_{2,S^l_m,S^l_m} \\ \left(1-\alpha_{1,S^l_m,S^l_m}\right)\gamma_{2,S^l_m,S^l_m} & \beta_{1,S^l_m,S^l_m}\eta_{2,S^l_m,S^l_m} & \left(1-\alpha_{1,S^l_m,S^l_m}\right)\left(1-\mu_{2,S^l_m,S^l_m}\right) & \beta_{1,S^l_m,S^l_m}1-\theta_{2,S^l_m,S^l_m}) \\ \alpha_{1,S^l_m,S^l_m}\gamma_{2,S^l_m,S^l_m} & \left(1-\beta_{1,S^l_m,S^l_m}\right)\eta_{2,S^l_m,S^l_m} & \alpha_{1,S^l_m,S^l_m}\left(1-\mu_{2,S^l_m,S^l_m}\right) & \left(1-\beta_{1,S^l_m,S^l_m}\right)\left(1-\theta_{2,S^l_m,S^l_m}\right) \end{bmatrix} \tag{18}$$

$$X_{m,k_{S^l_m}} = \left[P\left(g_{1,S^l_{m+1,1}}g_{2,S^l_{m,k_{S^l_m}}}\right) \quad P\left(d_{1,S^l_{m+1,1}}g_{2,S^l_{m,k_{S^l_m}}}\right) P\left(g_{1,S^l_{m+1,1}}d_{2,S^l_{m,k_{S^l_m}}}\right) \quad P\left(g_{1,S^l_{m+1,1}}d_{2,S^l_{m,k_{S^l_m}}}\right) \right] \tag{19}$$

$$A_{m,k_{S^l_m}} = \begin{bmatrix} \left(1-\alpha_{1,S^l_m,S^l_{m+1}}\right)\left(1-\gamma_{2,S^l_m,S^l_m}\right) & \beta_{1,S^l_m,S^l_{m+1}}\left(1-\eta_{2,S^l_m,S^l_m}\right) & \left(1-\alpha_{1,S^l_m,S^l_{m+1}}\right)\mu_{2,S^l_m,S^l_m} & \beta_{1,S^l_m,S^l_{m+1}}\theta_{2,S^l_m,S^l_m} \\ \alpha_{1,S^l_m,S^l_{m+1}}\left(1-\gamma_{2,S^l_m,S^l_m}\right) & \left(1-\beta_{1,S^l_m,S^l_{m+1}}\right)\left(1-\eta_{2,S^l_m,S^l_m}\right) & \alpha_{1,S^l_m,S^l_{m+1}}\mu_{2,S^l_m,S^l_m} & \left(1-\beta_{1,S^l_m,S^l_{m+1}}\right)\theta_{2,S^l_m,S^l_m} \\ \left(1-\alpha_{1,S^l_m,S^l_{m+1}}\right)\gamma_{2,S^l_m,S^l_m} & \beta_{1,S^l_m,S^l_{m+1}}\eta_{2,S^l_m,S^l_m} & \left(1-\alpha_{1,S^l_m,S^l_{m+1}}\right)\left(1-\mu_{2,S^l_m,S^l_m}\right) & \beta_{1,S^l_m,S^l_{m+1}}1-\theta_{2,S^l_m,S^l_m}) \\ \alpha_{1,S^l_m,S^l_{m+1}}\gamma_{2,S^l_m,S^l_m} & \left(1-\beta_{1,S^l_m,S^l_{m+1}}\right)\eta_{2,S^l_m,S^l_m} & \alpha_{1,S^l_m,S^l_{m+1}}\left(1-\mu_{2,S^l_m,S^l_m}\right) & \left(1-\beta_{1,S^l_m,S^l_{m+1}}\right)\left(1-\theta_{2,S^l_m,S^l_m}\right) \end{bmatrix} \tag{20}$$

By summing up the elements in $X_{i,j}$, we find that

$$\begin{aligned} &P\left(g_{1,S^l_{m,j+1}}g_{2,S^l_{m,j}}\right) + P\left(d_{1,S^l_{m,j+1}}g_{2,S^l_{m,j}}\right) + P\left(g_{1,S^l_{m,j+1}}d_{2,S^l_{m,j}}\right) \\ &\quad + P\left(g_{1,S^l_{m,j+1}}d_{2,S^l_{m,j}}\right) \\ &= P\left(g_{1,S^l_{m,j+2}}g_{2,S^l_{m,j+1}}\right) + P\left(d_{1,S^l_{m,j+2}}g_{2,S^l_{m,j+1}}\right) \\ &\quad + P\left(g_{1,S^l_{m,j+2}}d_{2,S^l_{m,j+1}}\right) + P\left(g_{1,S^l_{m,j+2}}d_{2,S^l_{m,j+1}}\right) \end{aligned} \tag{25}$$

which leads to

$$\begin{aligned} &P\left(g_{1,S^l_{m,j+1}}g_{2,S^l_{m,j}}\right) + P\left(d_{1,S^l_{m,j+1}}g_{2,S^l_{m,j}}\right) + P\left(g_{1,S^l_{m,j+1}}d_{2,S^l_{m,j}}\right) \\ &\quad + P\left(g_{1,S^l_{m,j+1}}d_{2,S^l_{m,j}}\right) = \frac{1}{K}. \end{aligned} \tag{26}$$

The final quality can be obtained from (23)–(26)

$$P\left(g^l\right) = \sum_{m=1}^{n}\sum_{j=1}^{k_{s^l_i}} P\left(g_{2,s^l_m,j}\right). \tag{27}$$

$$P\left(g_{\text{bp}}\right) = \frac{\sum_{m=1}^{n}\left(k_{S_m^l} - 1\right)\left(1 - \gamma_{2,S_m^l,S_m^l}\right) + \sum_{m=2}^{n}\left(1 - \gamma_{2,S_{m-1}^l,S_m^l}\right) + \left(1 - \gamma_{2,S_n^l,S_1^l}\right)}{\sum_{m=1}^{n} k_{S_m^l}} \tag{28}$$

$$\alpha'_{i,S_m^l,S_m^l} = \frac{P\left(g'_{i-1,S_m^l} g_{i,S_m^l}, j\right)\gamma_{i,S_m^l,S_m^l} + P\left(d'_{i-1,S_m^l} g_{i,S_m^l}, j\right)\eta_{i,S_m^l,S_m^l}}{P\left(g'_{i-1,S_m^l} g_{i,S_m^l}, j\right) + P\left(d'_{i-1,S_m^l} g_{i,S_m^l}, j\right)} \tag{29}$$

$$\alpha''_{i,S_m^l,S_m^l} = \frac{P\left(g'_{i-1,S_m^l} g_{i,S_m^l}, 1\right)\gamma_{i,S_m^l,S_m^l} + P\left(d'_{i-1,S_m^l} g_{i,S_m^l}, 1\right)\eta_{i,S_m^l,S_m^l}}{P\left(g'_{i-1,S_m^l} g_{i,S_m^l}, 1\right) + P\left(d'_{i-1,S_m^l} g_{i,S_m^l}, 1\right)} \tag{30}$$

$$\alpha'_{i,S_m^l,S_{m+1}^l} = \frac{P\left(g'_{i-1,S_{m+1}^l} g_{i-1,S_m^l}, k_{S_m^l}\right)\gamma_{i,S_m^l,S_{m+1}^l} + P\left(d'_{i-1,S_{m+1}^l} g_{i,S_m^l}, k_{S_m^l}\right)\eta_{i,S_m^l,S_{m+1}^l}}{P\left(g'_{i-1,S_{m+1}^l} g_{i-1,S_m^l}, k_{S_m^l}\right) + P\left(d'_{i-1,S_{m+1}^l} g_{i,S_m^l}, k_{S_m^l}\right)} \tag{31}$$

$$\beta'_{i,S_m^l,S_m^l} = \frac{P\left(g'_{i-1,S_m^l} d_{i,S_m^l}, j\right)\mu_{i,S_m^l,S_m^l} + P\left(d'_{i-1,S_m^l} d_{i,S_m^l}, j\right)\theta_{i,S_m^l,S_m^l}}{P\left(g'_{i-1,S_m^l} d_{i,S_m^l}, j\right) + P\left(d'_{i-1,S_m^l} d_{i,S_m^l}, j\right)} \tag{32}$$

$$\beta''_{i,S_m^l,S_m^l} = \frac{P\left(g'_{i-1,S_m^l} d_{i,S_m^l}, 1\right)\mu_{i,S_m^l,S_m^l} + P\left(d'_{i-1,S_m^l} d_{i,S_m^l}, 1\right)\theta_{i,S_m^l,S_m^l}}{P\left(g'_{i-1,S_m^l} d_{i,S_m^l}, 1\right) + P\left(d'_{i-1,S_m^l} d_{i,S_m^l}, 1\right)} \tag{33}$$

$$\beta'_{i,S_m^l,S_{m+1}^l} = \frac{P\left(g_{i-1,S_{m+1}^l} d_{i,S_m^l}, k_{S_m^l}\right)\mu_{i,S_m^l,S_{m+1}^l} + P\left(d_{i-1,S_{m+1}^l} d_{i,S_m^l}, k_{S_m^l}\right)\theta_{i,S_m^l,S_{m+1}^l}}{P\left(g_{i-1,S_{m+1}^l} d_{i,S_m^l}, k_{S_m^l}\right) + P\left(d_{i-1,S_{m+1}^l} d_{i,S_m^l}, k_{S_m^l}\right)} \tag{34}$$

If we ignore quality propagation in the system, namely, $\eta_{2,S_m^l,S_m^l} = \gamma_{2,S_m^l,S_m^l}$, $\eta_{2,S_{m-1}^l,S_m^l} = \gamma_{2,S_{m-1}^l,S_m^l}$, and $\eta_{2,S_n^l,S_1^l} = \gamma_{2,S_n^l,S_1^l}$, then the probability of producing a product with good quality could be (28), as shown at the top of this page, which is consistent with conclusion (26) made in [9].

C. Proposed Markov Model for Multiproduct-Multistage Systems

Based on the model for single-product-multistage systems that focuses on quality propagation and the model for multiproduct-two-stage systems that focuses on product flexibility, we derive the general model for M^{3}Ss. The corresponding probabilities are: α'_{i,S_m^l,S_m^l}, α''_{i,S_m^l,S_m^l}, $\alpha'_{i,S_m^l,S_{m+1}^l}$, β'_{i,S_m^l,S_m^l}, β''_{i,S_m^l,S_m^l}, $\beta'_{i,S_m^l,S_{m+1}^l}$, γ_{i,S_m^l,S_m^l}, $\gamma_{i,S_m^l,S_{m+1}^l}$, η_{i,S_m^l,S_m^l}, η_{i,S_m^l,S_{m+1}^l}, μ_{i,S_m^l,S_m^l}, μ_{i,S_m^l,S_{m+1}^l}, θ_{i,S_m^l,S_m^l}, and $\theta_{i,S_m^l,S_{m+1}^l}$, where $m = 1, 2, \ldots, r-1$.

Especially when $m = n$, it means a batch cycle has been finished and the system will enter another production cycle. Under this situation, $\alpha'_{i,S_m^l,S_{m+1}^l}$, for example, will be denoted as α'_{i,S_n^l,S_1^l} and so as for $\beta'_{i,S_m^l,S_{m+1}^l}$, $\gamma_{i,S_m^l,S_{m+1}^l}$, η_{i,S_m^l,S_{m+1}^l}, μ_{i,S_m^l,S_{m+1}^l}, and $\theta_{i,S_m^l,S_{m+1}^l}$. α'_{i,S_m^l,S_m^l}, α''_{i,S_m^l,S_m^l}, $\alpha'_{i,S_m^l,S_{m+1}^l}$, β'_{i,S_m^l,S_m^l}, β''_{i,S_m^l,S_m^l}, and $\beta'_{i,S_m^l,S_{m+1}^l}$ are obtained by applying iteration method and they can be expressed as follows, (29)–(34), as shown at the top of this page.

Similar to the matrices of transition probabilities that have been stated in Section IV-B, the transition probability matrices for the products in the first, middle, and last positions are (35)–(37), as shown at the top of the next page.

According to the definition of a Markov chain, we can get the probability of producing a product with good quality for M^{3}Ss. Especially when the probabilities satisfy Bernoulli distribution, we have $\alpha_{1,i,j} + \beta_{1,i,j} = 1$, $\gamma_{1,i,j} + \mu_{2,i,j} = 1$, $\eta_{1,i,j} + \theta_{2,i,j} = 1 (i = 1, 2, \ldots, n, j = 1, 2, \ldots, n)$ [8], [40], [45], [46], then the expression of the final quality for a n-product-r-stage system with sequence S^l can be obtained (38), as shown at the top of the next page.

V. Product Sequence Analysis

As indicated in the models for multiproduct-two-stage systems and multiproduct-multistage systems, different sequences involve different transition probabilities, which means the final quality might differ from each other under different sequences. Therefore, an appropriate sequence of multiple products could improve the final product quality. Furthermore, different sequence strategies, BP as $a, \cdots, a, b, \cdots b, c, \cdots c, \cdots, a, \cdots$, or SP as $a, b, c, \cdots a, b, c, \cdots$ also have impact on the quality.

A. Bernoulli Case

In order to simplify the analysis and make conclusions more explicit, we first focus on the quality properties regarding the sequence of multiproduct-two-stage systems under the Bernoulli case. The Bernoulli case is often employed in

$$A_{S^l_m,1}=\begin{bmatrix}\left(1-\alpha''_{i,S^l_m,S^l_m}\right)\left(1-\gamma_{i+1,S^l_{m-1},S^l_m}\right) & \beta''_{i,S^l_m,S^l_m}\left(1-\eta_{i+1,S^l_{m-1},S^l_m}\right) & \left(1-\alpha''_{i,S^l_m,S^l_m}\right)\mu_{i+1,S^l_{m-1},S^l_m} & \beta''_{i,S^l_m,S^l_m}\theta_{i+1,S^l_{m-1},S^l_m}\\ \alpha''_{i,S^l_m,S^l_m}\left(1-\gamma_{i+1,S^l_{m-1},S^l_m}\right) & \left(1-\beta''_{i,S^l_m,S^l_m}\right)\left(1-\eta_{i+1,S^l_{m-1},S^l_m}\right) & \alpha''_{i,S^l_m,S^l_m}\mu_{i+1,S^l_{m-1},S^l_m} & \left(1-\beta''_{i,S^l_m,S^l_m}\right)\theta_{i+1,S^l_{m-1},S^l_m}\\ \left(1-\alpha''_{i,S^l_m,S^l_m}\right)\gamma_{i+1,S^l_{m-1},S^l_m} & \beta''_{i,S^l_m,S^l_m}\eta_{i+1,S^l_{m-1},S^l_m} & \left(1-\alpha''_{i,S^l_m,S^l_m}\right)\left(1-\mu_{i+1,S^l_{m-1},S^l_m}\right) & \beta''_{i,S^l_m,S^l_m}\left(1-\theta_{i+1,S^l_{m-1},S^l_m}\right)\\ \alpha''_{i,S^l_m,S^l_m}\gamma_{i+1,S^l_{m-1},S^l_m} & \left(1-\beta''_{i,S^l_m,S^l_m}\right)\eta_{i+1,S^l_{m-1},S^l_m} & \left(1-\alpha''_{i,S^l_m,S^l_m}\right)\mu_{i+1,S^l_{m-1},S^l_m} & \left(1-\beta''_{i,S^l_m,S^l_m}\right)\left(1-\theta_{i+1,S^l_{m-1},S^l_m}\right)\end{bmatrix} \tag{35}$$

$$A_{S^l_m,j}=\begin{bmatrix}\left(1-\alpha'_{i,S^l_m,S^l_m}\right)\left(1-\gamma_{i+1,S^l_m,S^l_m}\right) & \beta'_{i,S^l_m,S^l_m}\left(1-\eta_{i+1,S^l_m,S^l_m}\right) & \left(1-\alpha'_{i,S^l_m,S^l_m}\right)\mu_{i+1,S^l_m,S^l_m} & \beta'_{i,S^l_m,S^l_m}\theta_{i+1,S^l_m,S^l_m}\\ \alpha'_{i,S^l_m,S^l_m}\left(1-\gamma_{i+1,S^l_m,S^l_m}\right) & \left(1-\beta'_{i,S^l_m,S^l_m}\right)\left(1-\eta_{i+1,S^l_m,S^l_m}\right) & \alpha'_{i,S^l_m,S^l_m}\mu_{i+1,S^l_m,S^l_m} & \left(1-\beta'_{i,S^l_m,S^l_m}\right)\theta_{i+1,S^l_m,S^l_m}\\ \left(1-\alpha'_{i,S^l_m,S^l_m}\right)\gamma_{i+1,S^l_m,S^l_m} & \beta'_{i,S^l_m,S^l_m}\eta_{i+1,S^l_m,S^l_m} & \left(1-\alpha'_{i,S^l_m,S^l_m}\right)\left(1-\mu_{i+1,S^l_m,S^l_m}\right) & \beta'_{i,S^l_m,S^l_m}\left(1-\theta_{i+1,S^l_m,S^l_m}\right)\\ \alpha'_{i,S^l_m,S^l_m}\gamma_{i+1,S^l_m,S^l_m} & \left(1-\beta'_{i,S^l_m,S^l_m}\right)\eta_{i+1,S^l_m,S^l_m} & \left(1-\alpha'_{i,S^l_m,S^l_m}\right)\mu_{i+1,S^l_m,S^l_m} & \left(1-\beta'_{i,S^l_m,S^l_m}\right)\left(1-\theta_{i+1,S^l_m,S^l_m}\right)\end{bmatrix} \tag{36}$$

$$A_{S^l_m,k_{S^l_m}}=\begin{bmatrix}\left(1-\alpha'_{i,S^l_m,S^l_{m+1}}\right)\left(1-\gamma_{i+1,S^l_m,S^l_m}\right) & \beta'_{i,S^l_m,S^l_{m+1}}\left(1-\eta_{i+1,S^l_m,S^l_m}\right) & \left(1-\alpha'_{i,S^l_m,S^l_{m+1}}\right)\mu_{i+1,S^l_m,S^l_m} & \beta'_{i,S^l_m,S^l_{m+1}}\theta_{i+1,S^l_m,S^l_m}\\ \alpha'_{i,S^l_{m+1},S^l_m}\left(1-\gamma_{i+1,S^l_m,S^l_m}\right) & \left(1-\beta'_{i,S^l_m,S^l_{m+1}}\right)\left(1-\eta_{i+1,S^l_m,S^l_m}\right) & \alpha'_{i,S^l_m,S^l_{m+1}}\mu_{i+1,S^l_m,S^l_m} & \left(1-\beta'_{i,S^l_m,S^l_{m+1}}\right)\theta_{i+1,S^l_m,S^l_m}\\ \left(1-\alpha'_{i,S^l_m,S^l_{m+1}}\right)\gamma_{i+1,S^l_m,S^l_m} & \beta'_{i,S^l_m,S^l_{m+1}}\eta_{i+1,S^l_m,S^l_m} & \left(1-\alpha'_{i,S^l_m,S^l_{m+1}}\right)\left(1-\mu_{i+1,S^l_m,S^l_m}\right) & \beta'_{i,S^l_m,S^l_{m+1}}\left(1-\theta_{i+1,S^l_m,S^l_m}\right)\\ \alpha'_{i,S^l_m,S^l_{m+1}}\gamma_{i+1,S^l_m,S^l_m} & \left(1-\beta'_{i,S^l_m,S^l_{m+1}}\right)\eta_{i+1,S^l_m,S^l_m} & \left(1-\alpha'_{i,S^l_m,S^l_{m+1}}\right)\mu_{i+1,S^l_m,S^l_m} & \left(1-\beta'_{i,S^l_m,S^l_{m+1}}\right)\left(1-\theta_{i+1,S^l_m,S^l_m}\right)\end{bmatrix} \tag{37}$$

$$\begin{aligned}P(g_{\text{bp}}) =& \frac{\sum_{i=1}^{n}\left(k_{S^l_i}-1\right)\left[1-\gamma_{r,S^l_i,S^l_i}-\gamma_{r-1,S^l_i,S^l_i}\left(\eta_{r,S^l_i,S^l_i}-\gamma_{r,S^l_i,S^l_i}\right)-\cdots-\gamma_{3,S^l_i,S^l_i}\prod_{t=4}^{r}\left(\eta_{t,S^l_i,S^l_i}-\gamma_{t,S^l_i,S^l_i}\right)\right]}{\sum_{i=1}^{n}k_{S^l_i}}\\ &-\frac{\sum_{i=1}^{n}\left(k_{S^l_i}-1\right)\left[\left(1-\alpha_{1,S^l_i,S^l_i}\right)\gamma_{2,S^l_i,S^l_i}\prod_{t=3}^{r}\left(\eta_{t,S^l_i,S^l_i}-\gamma_{t,S^l_i,S^l_i}\right)+\alpha_{1,S^l_i,S^l_i}\eta_{2,S^l_i,S^l_i}\prod_{t=3}^{r}\left(\eta_{t,S^l_i,S^l_i}-\gamma_{t,S^l_i,S^l_i}\right)\right]}{\sum_{i=1}^{n}k_{S^l_i}}\\ &+\frac{\sum_{i=2}^{n}\left[1-\gamma_{r,S^l_{i-1},S^l_i}-\gamma_{r-1,S^l_{i-1},S^l_i}\left(\eta_{r,S^l_{i-1},S^l_i}-\gamma_{r,S^l_{i-1},S^l_i}\right)-\cdots-\gamma_{3,S^l_{i-1},S^l_i}\prod_{t=4}^{r}\left(\eta_{t,S^l_{i-1},S^l_i}-\gamma_{t,S^l_{i-1},S^l_i}\right)\right]}{\sum_{i=1}^{n}k_{S^l_i}}\\ &-\frac{\sum_{i=2}^{n}\left[\left(1-\alpha_{1,S^l_{i-1},S^l_i}\right)\gamma_{2,S^l_{i-1},S^l_i}\prod_{t=3}^{r}\left(\eta_{t,S^l_{i-1},S^l_i}-\gamma_{t,S^l_{i-1},S^l_i}\right)+\alpha_{1,S^l_{i-1},S^l_i}\eta_{2,S^l_{i-1},S^l_i}\prod_{t=3}^{r}\left(\eta_{t,S^l_{i-1},S^l_i}-\gamma_{t,S^l_{i-1},S^l_i}\right)\right]}{\sum_{i=1}^{n}k_{S^l_i}}\\ &+\frac{\left[1-\gamma_{r,S^l_n,S^l_1}-\gamma_{r-1,S^l_n,S^l_1}\left(\eta_{r,S^l_n,S^l_1}-\gamma_{r,S^l_n,S^l_1}\right)-\cdots-\gamma_{3,S^l_n,S^l_1}\prod_{t=4}^{r}\left(\eta_{t,S^l_n,S^l_1}-\gamma_{t,S^l_n,S^l_1}\right)\right]}{\sum_{i=1}^{n}k_{S^l_i}}\\ &-\frac{\left[\left(1-\alpha_{1,S^l_n,S^l_1}\right)\gamma_{2,S^l_n,S^l_1}\prod_{t=3}^{r}\left(\eta_{t,S^l_n,S^l_1}-\gamma_{t,S^l_n,S^l_1}\right)+\alpha_{1,S^l_n,S^l_1}\eta_{2,S^l_n,S^l_1}\prod_{t=3}^{r}\left(\eta_{t,S^l_n,S^l_1}-\gamma_{t,S^l_n,S^l_1}\right)\right]}{\sum_{i=1}^{n}k_{S^l_i}}\end{aligned} \tag{38}$$

$$\begin{aligned}P(g_{\text{bp}}) =& \frac{\sum_{m=1}^{n}\left(k_{S^l_m}-1\right)\left(1-\alpha_{1,S^l_m,S^l_m}\eta_{2,S^l_m,S^l_m}+\alpha_{1,S^l_m,S^l_m}\gamma_{2,S^l_m,S^l_m}-\gamma_{2,S^l_m,S^l_m}\right)}{\sum_{m=1}^{n}k_{S^l_m}}\\ &+\frac{\sum_{m=2}^{n}\left(1-\alpha_{1,S^l_{m-1},S^l_m}\eta_{2,S^l_{m-1},S^l_m}+\alpha_{1,S^l_{m-1},S^l_m}\gamma_{2,S^l_{m-1},S^l_m}-\gamma_{2,S^l_{m-1},S^l_m}\right)}{\sum_{m=1}^{n}k_{S^l_m}}\\ &+\frac{\left(1-\alpha_{1,S^l_n,S^l_1}\eta_{2,S^l_n,S^l_1}+\alpha_{1,S^l_n,S^l_1}\gamma_{2,S^l_n,S^l_1}-\gamma_{2,S^l_n,S^l_1}\right)}{\sum_{m=1}^{n}k_{S^l_m}}\end{aligned} \tag{39}$$

$$\begin{aligned}P(g_{\text{sp}}) =& \frac{\sum_{m=2}^{n}\left(1-\alpha_{1,S^l_{m-1},S^l_m}\eta_{2,S^l_{m-1},S^l_m}+\alpha_{1,S^l_{m-1},S^l_m}\gamma_{2,S^l_{m-1},S^l_m}-\gamma_{1,S^l_{m-1},S^l_m}\right)}{n}\\ &+\frac{\left(1-\alpha_{1,S^l_n,S^l_1}\eta_{2,S^l_n,S^l_1}+\alpha_{1,S^l_n,S^l_1}\gamma_{2,S^l_n,S^l_1}-\gamma_{1,S^l_n,S^l_1}\right)}{n}\end{aligned} \tag{40}$$

quality analysis and assumes that the system quality follows a Bernoulli distribution: $\alpha_{1,i,j}+\beta_{1,i,j}=1$, $\gamma_{1,i,j}+\mu_{2,i,j}=1$, and $\eta_{1,i,j}+\theta_{2,i,j}=1 (i=1,2,\ldots,n, j=1,2,\ldots,n)$. Accordingly, we obtain the probability that the system produces good products under BP $P(g_{\text{bp}})$ and SP $P(g_{\text{sp}})$, respectively, (39) and (40), as shown at the top of this page.

$$P_{\mathrm{bp}}(g_{S^1}) = \frac{(k_a-1)[1-\gamma_{2,a,a}+\alpha_{1,a,a}(\gamma_{2,a,a}-\eta_{2,a,a})]}{k_a+k_b+k_c} + \frac{(k_b-1)[1-\gamma_{2,b,b}+\alpha_{1,b,b}(\gamma_{2,b,b}-\eta_{2,b,b})]}{k_a+k_b+k_c} + \frac{(k_c-1)[1-\gamma_{2,c,c}+\alpha_{1,c,c}(\gamma_{2,c,c}-\eta_{2,c,c})]}{k_a+k_b+k_c} + \frac{1-\gamma_{2,c,a}+\alpha_{1,c,a}(\gamma_{2,c,a}-\eta_{2,c,a})+1-\gamma_{2,b,c}+\alpha_{1,b,c}(\gamma_{2,b,c}-\eta_{2,b,c})}{k_a+k_b+k_c} + \frac{1-\gamma_{2,a,b}+\alpha_{1,a,b}(\gamma_{2,a,b}-\eta_{2,a,b})}{k_a+k_b+k_c} \tag{41}$$

$$P_{\mathrm{bp}}(g_{S^2}) = \frac{(k_a-1)\left[1-\gamma_{2,a,a}+\alpha_{1,a,a}(\gamma_{2,a,a}-\eta_{2,a,a})\right]}{k_A+k_B+k_C} + \frac{(k_b-1)\left[1-\gamma_{2,b,b}+\alpha_{1,b,b}(\gamma_{2,b,b}-\eta_{2,b,b})\right]}{k_A+k_B+k_C} + \frac{(k_c-1)\left(1-\gamma_{2,c,c}+\alpha_{1,c,c}(\gamma_{2,c,c}-\eta_{2,c,c})\right]}{k_A+k_B+k_C} + \frac{1-\gamma_{2,b,a}+\alpha_{1,b,a}(\gamma_{2,b,a}-\eta_{2,b,a})+1-\gamma_{2,c,b}+\alpha_{1,c,b}(\gamma_{2,c,b}-\eta_{2,c,b})}{k_a+k_b+k_c} + \frac{1-\gamma_{2,a,c}+\alpha_{1,a,c}(\gamma_{2,a,c}-\eta_{2,a,c})}{k_a+k_b+k_c} \tag{42}$$

$$P_{\mathrm{sp}}(g_{S^1}) = \frac{1-\gamma_{2,c,a}+\alpha_{1,c,a}(\gamma_{2,c,a}-\eta_{2,c,a})+1-\gamma_{2,b,c}+\alpha_{1,b,c}(\gamma_{2,b,c}-\eta_{2,b,c})}{3} + \frac{1-\gamma_{2,a,b}+\alpha_{1,a,b}(\gamma_{2,a,b}-\eta_{2,a,b})}{3} \tag{43}$$

$$P_{\mathrm{sp}}(g_{S^2}) = \frac{1-\gamma_{2,b,a}+\alpha_{1,b,a}(\gamma_{2,b,a}-\eta_{2,b,a})+1-\gamma_{2,c,b}+\alpha_{1,c,b}(\gamma_{2,c,b}-\eta_{2,c,b})}{3} + \frac{1-\gamma_{2,a,c}+\alpha_{1,a,c}(\gamma_{2,a,c}-\eta_{2,a,c})}{3} \tag{44}$$

Here we first take a three-product-two-stage system as an example to study the quality properties of product sequence and then extend these properties to more general cases. Assume that three different products a, b, c are produced. There exist two different sequences: one is S^1: $a \to b \to c$, the other is S^2: $a \to c \to b$. The corresponding probabilities are

$$\alpha_{1,a,b}, \alpha_{1,b,c}, \alpha_{1,c,a}, \alpha_{1,a,c}, \alpha_{1,c,b}, \alpha_{1,b,a}, \alpha_{1,a,a}, \alpha_{1,b,b}, \alpha_{1,c,c}$$
$$\gamma_{2,a,b}, \gamma_{2,b,c}, \gamma_{2,c,a}, \gamma_{2,a,c}, \gamma_{2,c,b}, \gamma_{2,b,a}, \gamma_{2,a,a}, \gamma_{2,b,b}, \gamma_{2,c,c}$$
$$\eta_{2,a,b}, \eta_{2,b,c}, \eta_{2,c,a}, \eta_{2,a,c}, \eta_{2,c,b}, \eta_{2,b,a}, \eta_{2,a,a}, \eta_{2,b,b}, \eta_{2,c,c}.$$

According to (39), the probabilities of producing a good product for sequence S^1 and S^2 under BP are expressed as $P(g^1_{\mathrm{bp}})$ and $P(g^2_{\mathrm{bp}})$, respectively, (41) and (42), as shown at the top of this page.

And the probabilities of producing a good product under SP for the two sequences can be obtained from (40) and expressed as $P(g^1_{\mathrm{sp}})$ and $P(g^2_{\mathrm{sp}})$, respectively, (43) and (44) as shown at the top of this page.

Comparing (41) with (42), we can draw Conclusion 1.

Conclusion 1: When BP is applied in M^{3}Ss, the best product sequence depends on the external rather than internal transition probabilities. By comparing the difference among the external transition probabilities under different sequences, we can find out the best product sequence.

Comparing (41) with (43) [or comparing (42) with (44)], we have Conclusion 2.

Conclusion 2: Best sequences under BP and SP in M^{3}Ss are consistent with each other, which means if sequence 1 is the best one under BP, then it's also the best one under SP.

Conclusions 1 and 2 can be extended to Conclusion 3 when n types of products are manufactured in the system.

Conclusion 3: Under the Bernoulli case, for an M^{3}S processing n different types of products, the best sequences under BP and SP are consistent with each other, which means $P(g^l_{\mathrm{bp}}) > P(g^m_{\mathrm{bp}}) \Leftrightarrow P(g^l_{\mathrm{sp}}) > P(g^m_{\mathrm{sp}})$. And the sequence satisfying the condition

$$\max \left\{ \sum_{m=2}^{n} \left(1-\alpha_{1,S^l_{m-1},S^l_m}\eta_{2,S^l_{m-1},S^l_m}+\alpha_{1,S^l_{m-1},S^l_m}\gamma_{2,S^l_{m-1},S^l_m} - \gamma_{2,S^l_{m-1},S^l_m}\right) + \left(1-\alpha_{1,S^l_n,S^l_1}\eta_{2,S^l_n,S^l_1}+\alpha_{1,S^l_n,S^l_1}\gamma_{2,S^l_n,S^l_1}-\gamma_{2,S^l_n,S^l_1}\right)\right\}$$

is the best sequence.

Proof of Conclusion 3 can be found in the Appendix.

B. Bernoulli Relaxation Case

Although the Bernoulli case is very similar to the real manufacturing conditions, it still seems strict to some extent. Thus, we slightly relax the Bernoulli case and extend the model to more general cases in practice. Under the Bernoulli relaxation case, the summation of failure probability and repair probability does not have to equal 1 but just to be close to 1. In other words

$$\delta_1 = \max\left\{\left|1-\alpha_{1,s_i^l,s_j^l}-\beta_{1,s_i^l,s_j^l}\right|\right\}$$
$$\delta_2 = \max\left\{\left|1-\gamma_{2,s_i^l,s_j^l}-\mu_{2,s_i^l,s_j^l}\right|\right\}$$
$$\delta_3 = \max\left\{\left|1-\eta_{2,s_i^l,s_j^l}-\theta_{2,s_i^l,s_j^l}\right|\right\}$$
$$\delta_{\max} = \max\{\delta_1, \delta_2, \delta_3\}$$

where $i = 1,2,\ldots,n, j = 1,2,\ldots,n$, and $0 \leq \delta_1, \delta_2, \delta_3, \delta_{\max} \ll 1$.

Conclusion 4: The probability of producing a good product for M^3Ss in a certain interval is decided by $\delta_{\max}$ and

$$\frac{1}{1+\delta_{\max}}\chi^l \leq P\left(g^l\right) \leq \frac{1}{1-\delta_{\max}}\chi^l \quad (45)$$

where for BP

$$\begin{aligned}\chi^l &= \chi^l_{\text{bp}} = P\left(g^l_{\text{bp}}\right)\\ &= \frac{\sum_{i=1}^n\left(k_{S_i^l}-1\right)\left(1-\alpha_{1,S_i^l,S_i^l}\eta_{S_i^l,S_i^l}+\alpha_{1,S_i^l,S_i^l}\gamma_{2,S_i^l,S_i^l}-\gamma_{2,S_i^l,S_i^l}\right)}{\sum_{i=1}^n k_{S_i^l}}\\ &+\frac{\sum_{i=2}^n\left(1-\alpha_{1,S_{i-1}^l,S_i^l}\eta_{2,S_{i-1}^l,S_i^l}+\alpha_{1,S_{i-1}^l,S_i^l}\gamma_{2,S_{i-1}^l,S_i^l}-\gamma_{2,S_{i-1}^l,S_i^l}\right)}{\sum_{i=1}^n k_{S_i^l}}\\ &+\frac{\left(1-\alpha_{1,S_n^l,S_1^l}\eta_{2,S_n^l,S_1^l}+\alpha_{1,S_n^l,S_1^l}\gamma_{2,S_n^l,S_1^l}-\gamma_{2,S_n^l,S_1^l}\right)}{\sum_{i=1}^n k_{S_i^l}}\end{aligned} \quad (46)$$

and for SP

$$\begin{aligned}\chi^l &= \chi^l_{\text{sp}} = P\left(g^l_{ss}\right)\\ &= \frac{\sum_{i=2}^n\left(1-\alpha_{1,S_{i-1}^l,S_i^l}\eta_{2,S_{i-1}^l,S_i^l}+\alpha_{1,S_{i-1}^l,S_i^l}\gamma_{2,S_{i-1}^l,S_i^l}-\gamma_{2,S_{i-1}^l,S_i^l}\right)}{n}\\ &+\frac{\left(1-\alpha_{1,S_n^l,S_1^l}\eta_{2,S_n^l,S_1^l}+\alpha_{1,S_n^l,S_1^l}\gamma_{2,S_n^l,S_1^l}-\gamma_{2,S_n^l,S_1^l}\right)}{n}.\end{aligned} \quad (47)$$

Proof of Conclusion 4 can be found in the Appendix.

Conclusion 5: Under Bernoulli relaxation case, when $0 < \delta_{\max} < [(\chi^l_{\text{sp}}-\chi^m_{\text{sp}})/(\chi^l_{\text{sp}}+\chi^m_{\text{sp}})] = [(\chi^l_{\text{bp}}-\chi^m_{\text{bp}})/(\chi^l_{\text{bp}}+\chi^m_{\text{bp}})]$, we still have $P(g^l_{\text{bp}}) > P(g^m_{\text{bp}}) \Leftrightarrow P(g^l_{\text{sp}}) > P(g^m_{\text{sp}})$.

Proof of Conclusion 5 can be found in the Appendix.

Actually not only when $0 < \delta_{\max} < [(\chi^l_{\text{sp}}-\chi^m_{\text{sp}})/(\chi^l_{\text{sp}}+\chi^m_{\text{sp}})] = [(\chi^l_{\text{bp}}-\chi^m_{\text{bp}})/(\chi^l_{\text{bp}}+\chi^m_{\text{bp}})]$ but also under most Bernoulli relaxation cases, the conclusion $P(g^l_{\text{bp}}) > P(g^m_{\text{bp}}) \Leftrightarrow P(g^l_{\text{sp}}) > P(g^m_{\text{sp}})$ holds. In order to verify it, extensive numerical experiments have been carried out. We assume that $\delta_{\max} \leq 0.2$ and $\alpha \in [0, 0.2]$, $\beta \in [0.8, 1]$, $\gamma \in [0, 0.2]$, $\mu \in [0.8, 1]$, $\eta \in [0, 1]$, and $\theta \in [0, 1]$ [9, 11]. Corresponding transition probabilities under certain sequences are randomly generated within their intervals and the results of the final quality are estimated by applying the model for multiproduct-two-stage systems. When there are more than three types of products, alternative sequences can be chosen to improve quality. Here we increase the number of product types from 3 to 6 and the number of possible sequences increases from 2 to 120. For the given number of product types, make comparisons between the results of final quality under different sequences and each comparison is based on 1000 times of numerical experiments. The results show that the probability that $P(g^l_{\text{bp}}) > P(g^m_{\text{bp}}) \Leftrightarrow P(g^l_{\text{sp}}) > P(g^m_{\text{sp}})$ holds is around 97% when the number of product types ranges from 3 to 6.

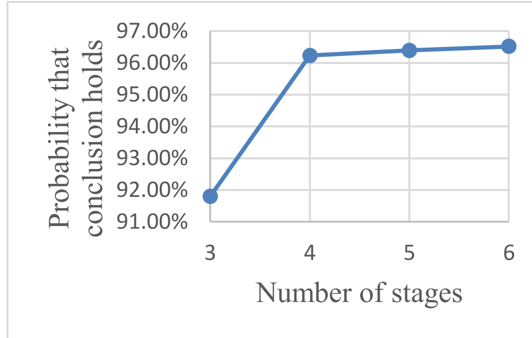

Fig. 4. Probability that conclusion holds under Bernoulli relaxation case.

When there are three types of products, the probability that the conclusion $P(g^l_{\text{bp}}) > P(g^m_{\text{bp}}) \Leftrightarrow P(g^l_{\text{sp}}) > P(g^m_{\text{sp}})$ holds is about 91%. But when the types of products increase to 4–6, the probabilities are all between 96% and 97% (Fig. 4).

So it is reasonable to conclude that in practice, the best sequences under BP and SP in M^3Ss are consistent with each other.

VI. Quality Improvement Analysis

A. Sensitivity Analysis

The product quality for a multistage system depends on the quality failure probabilities γ_i, η_i and quality repair probabilities μ_i, θ_i. Changes of theses parameters can lead to the improvement of quality $P(g_i)$. It is necessary to find out which parameter could bring about the largest quality improvement. Sensitivity analysis of $P(g_i)$ with respect to γ_i, η_i, μ_i, and θ_i can help figure out this question.

For the sensitivity analysis, we change only one parameter and while the others remain unchanged. Accordingly, the changed parameters and probabilities are γ_i', μ_i', η_i', θ_i', and $P_{\gamma_i}(g_i)$, $P_{\mu_i}(g_i)$, $P_{\eta_i}(g_i)$, $P_{\theta_i}(g_i)$, respectively. Then the sensitivity of $P(g_i)$ with respect to γ_i, μ_i, η_i, and θ_i could be written as

$$S_{\gamma_i} = \frac{\left|P_{\gamma_i}(g_i)-P(g_i)\right|/P(g_i)}{\left|\gamma_i'-\gamma_i\right|/\gamma_i} \quad (48)$$

$$S_{\mu_i} = \frac{\left|P_{\mu_i}(g_i)-P(g_i)\right|/P(g_i)}{\left|\mu_i'-\mu_i\right|/\mu_i} \quad (49)$$

$$S_{\eta_i} = \frac{\left|P_{\eta_i}(g_i)-P(g_i)\right|/P(g_i)}{\left|\eta_i'-\eta_i\right|/\eta_i} \quad (50)$$

$$S_{\theta_i} = \frac{\left|P_{\theta_i}(g_i)-P(g_i)\right|/P(g_i)}{\left|\theta_i'-\theta_i\right|/\theta_i}. \quad (51)$$

$$
\begin{aligned}
P(g_4) = {} & \frac{\left(k_{S_1^l}-1\right)\left[1-\gamma_{4,S_1^l,S_1^l}-\gamma_{3,S_1^l,S_1^l}\left(\eta_{4,S_1^l,S_1^l}-\gamma_{4,S_1^l,S_1^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} \\
& - \frac{\left(k_{S_1^l}-1\right)\left[\left(1-\alpha_{1,S_1^l,S_1^l}\right)\gamma_{2,S_1^l,S_1^l}\left(\eta_{3,S_1^l,S_1^l}-\gamma_{3,S_1^l,S_1^l}\right)\left(\eta_{4,S_1^l,S_1^l}-\gamma_{4,S_1^l,S_1^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} \\
& - \frac{\left(k_{S_1^l}-1\right)\left[\alpha_{1,S_1^l,S_1^l}\eta_{2,S_1^l,S_1^l}\left(\eta_{3,S_1^l,S_1^l}-\gamma_{3,S_1^l,S_1^l}\right)\left(\eta_{4,S_1^l,S_1^l}-\gamma_{4,S_1^l,S_1^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} \\
& + \frac{\left(k_{S_2^l}-1\right)\left[1-\gamma_{4,S_2^l,S_2^l}-\gamma_{3,S_2^l,S_2^l}\left(\eta_{4,S_2^l,S_2^l}-\gamma_{4,S_2^l,S_2^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} \\
& - \frac{\left(k_{S_2^l}-1\right)\left[\left(1-\alpha_{1,S_2^l,S_2^l}\right)\gamma_{2,S_2^l,S_2^l}\left(\eta_{3,S_2^l,S_2^l}-\gamma_{3,S_2^l,S_2^l}\right)\left(\eta_{4,S_2^l,S_2^l}-\gamma_{4,S_2^l,S_2^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} \\
& - \frac{\left(k_{S_2^l}-1\right)\left[\alpha_{1,S_2^l,S_2^l}\eta_{2,S_2^l,S_2^l}\left(\eta_{3,S_2^l,S_2^l}-\gamma_{3,S_2^l,S_2^l}\right)\left(\eta_{4,S_2^l,S_2^l}-\gamma_{4,S_2^l,S_2^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} + \frac{\left[1-\gamma_{4,S_2^l,S_1^l}-\gamma_{3,S_2^l,S_1^l}\left(\eta_{4,S_2^l,S_1^l}-\gamma_{4,S_2^l,S_1^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} \\
& - \frac{\left[\left(1-\alpha_{1,S_2^l,S_1^l}\right)\gamma_{2,S_2^l,S_1^l}\left(\eta_{3,S_2^l,S_1^l}-\gamma_{3,S_2^l,S_1^l}\right)\left(\eta_{4,S_2^l,S_1^l}-\gamma_{4,S_2^l,S_1^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} \\
& - \frac{\left[\alpha_{1,S_2^l,S_1^l}\eta_{2,S_2^l,S_1^l}\left(\eta_{3,S_2^l,S_1^l}-\gamma_{3,S_2^l,S_1^l}\right)\left(\eta_{4,S_2^l,S_1^l}-\gamma_{4,S_2^l,S_1^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} + \frac{\left[1-\gamma_{4,S_1^l,S_2^l}-\gamma_{3,S_1^l,S_2^l}\left(\eta_{4,S_1^l,S_2^l}-\gamma_{4,S_1^l,S_2^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} \\
& - \frac{\left[\left(1-\alpha_{1,S_1^l,S_2^l}\right)\gamma_{2,S_1^l,S_2^l}\left(\eta_{3,S_1^l,S_2^l}-\gamma_{3,S_1^l,S_2^l}\right)\left(\eta_{4,S_1^l,S_2^l}-\gamma_{4,S_1^l,S_2^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}} \\
& - \frac{\left[\alpha_{1,S_1^l,S_2^l}\eta_{2,S_1^l,S_2^l}\left(\eta_{3,S_1^l,S_2^l}-\gamma_{3,S_1^l,S_2^l}\right)\left(\eta_{4,S_1^l,S_2^l}-\gamma_{4,S_1^l,S_2^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}}
\end{aligned} \tag{55}
$$

Especially, when $i = 1$, the sensitivity analysis of $P(g_1)$ with respect to α_1 and β_1 is needed. According to [41], we can obtain

$$P(g_1) = \frac{\beta_1}{\alpha_1 + \beta_1}. \tag{52}$$

Assume the changed parameters are α_1', β_1' and $P_{\alpha_1}(g_1)$, $P_{\beta_1}(g_1)$. Then the sensitivity of $P(g_1)$ with respect to α_1 and β_1 would be

$$S_{\alpha_1} = \frac{\left|P_{\alpha_1}(g_1) - P(g_1)\right|/P(g_1)}{\left|\alpha_1' - \alpha_1\right|/\alpha_1} \tag{53}$$

$$S_{\beta_1} = \frac{\left|P_{\beta_1}(g_1) - P(g_1)\right|/P(g_1)}{\left|\beta_1' - \beta_1\right|/\beta_1}. \tag{54}$$

The parameter with $\max(S_{\gamma_i}, S_{\mu_i}, S_{\eta_i}, S_{\theta_i})$ when $i \geq 2$ or $\max(S_{\alpha_1}, S_{\beta_1})$ when $i = 1$ is the most sensitive and has the largest impact on $P(g_i)$.

B. Bottleneck Analysis

For M^{3}Ss, the final quality is the function of internal and external failure probabilities and repair probabilities. Because of product variety, a certain set of failure or repair probabilities contains both internal and external probabilities. For example, for quality failure probabilities with respect to good incoming parts γ_i, there are internal probabilities $\gamma_{i,S_m^l,S_m^l} (m = 1, 2, \ldots, n)$ and external probabilities $\gamma_{i,S_{m-1}^l,S_m^l} (m = 2, 3, \ldots, n)$. Then there are several kinds of quality bottlenecks, namely, internal and external quality bottleneck for γ (IQB $-\, \gamma$ and EQB $-\, \gamma$), internal and external quality bottleneck for μ (IQB $-\, \mu$ and EQB $-\, \mu$), internal and external quality bottleneck for η (IQB $-\, \eta$ and EQB $-\, \eta$), and internal and external quality bottleneck for θ (IQB $-\, \theta$ and EQB $-\, \theta$). They are defined as follows.

Definition 1: For $i \neq j$, if $|[\partial P(g_n)/\partial \gamma_{i,S_m^l,S_m^l}]| > |[\partial P(g_n)/\partial \gamma_{j,S_m^l,S_m^l}]|$, then under sequence S^l, γ_{i,S_m^l,S_m^l} is the IQB $-\, \gamma$ for product S_m^l, which means when producing product S_m^l, stage i is the quality bottleneck with respect to failure probabilities for good incoming parts.

Definition 2: For $i \neq j$, if $|[\partial P(g_n)/\partial \mu_{i,S_m^l,S_m^l}]| > |[\partial P(g_n)/\partial \mu_{j,S_m^l,S_m^l}]|$, then under sequence S^l, μ_{i,S_m^l,S_m^l} is the IQB $-\, \mu$ for product S_m^l, which means when producing product S_m^l, stage i is the quality bottleneck with respect to repair probabilities with defective incoming parts.

Definition 3: For $i \neq j$, if $|[\partial P(g_n)/\partial \eta_{i,S_m^l,S_m^l}]| > |[\partial P(g_n)/\partial \eta_{j,S_m^l,S_m^l}]|$, then under sequence S^l, η_{i,S_m^l,S_m^l} is the IQB $-\, \eta$ for product S_m^l, which means when producing product S_m^l, stage i is the quality bottleneck with respect to failure probabilities with defective incoming parts.

Definition 4: For $i \neq j$, if $|[\partial P(g_n)/\partial \theta_{i,S_m^l,S_m^l}]| > |[\partial P(g_n)/\partial \theta_{j,S_m^l,S_m^l}]|$, then under sequence S^l, θ_{i,S_m^l,S_m^l} is the IQB $-\, \theta$ for product S_m^l, which means when producing

$$\left|\frac{\partial P(g_4)}{\partial \gamma_{4,S_1^l,S_1^l}}\right| = \left|\frac{\left(k_{S_1^l}-1\right)\left\{\left[\left(1-\alpha_{1,S_1^l,S_1^l}\right)\left(\eta_{2,S_1^l,S_1^l}-\gamma_{2,S_1^l,S_1^l}\right)+1-\eta_{2,S_1^l,S_1^l}\right]\left(\eta_{3,S_1^l,S_1^l}-\gamma_{3,S_1^l,S_1^l}\right)+1-\eta_{3,S_1^l,S_1^l}\right\}}{k_{S_1^l}+k_{S_2^l}}\right| \tag{56}$$

$$\left|\frac{\partial P(g_4)}{\partial \gamma_{4,S_2^l,S_2^l}}\right| = \left|\frac{\left(k_{S_2^l}-1\right)\left\{\left[\left(1-\alpha_{1,S_2^l,S_2^l}\right)\left(\eta_{2,S_2^l,S_2^l}-\gamma_{2,S_2^l,S_2^l}\right)+1-\eta_{2,S_2^l,S_2^l}\right]\left(\eta_{3,S_2^l,S_2^l}-\gamma_{3,S_2^l,S_2^l}\right)+1-\eta_{3,S_2^l,S_2^l}\right\}}{k_{S_1^l}+k_{S_2^l}}\right| \tag{57}$$

$$\left|\frac{\partial P(g_4)}{\partial \gamma_{3,S_1^l,S_1^l}}\right| = \left|\frac{\left(k_{S_1^l}-1\right)\left[\left(1-\alpha_{1,S_1^l,S_1^l}\right)\left(\eta_{2,S_1^l,S_1^l}-\gamma_{2,S_1^l,S_1^l}\right)+1-\eta_{2,S_1^l,S_1^l}\right]\left(\eta_{4,S_1^l,S_1^l}-\gamma_{4,S_1^l,S_1^l}\right)}{k_{S_1^l}+k_{S_2^l}}\right| \tag{58}$$

$$\left|\frac{\partial P(g_4)}{\partial \gamma_{3,S_2^l,S_2^l}}\right| = \left|\frac{\left(k_{S_2^l}-1\right)\left[\left(1-\alpha_{1,S_2^l,S_2^l}\right)\left(\eta_{2,S_2^l,S_2^l}-\gamma_{2,S_2^l,S_2^l}\right)+1-\eta_{2,S_2^l,S_2^l}\right]\left(\eta_{4,S_2^l,S_2^l}-\gamma_{4,S_2^l,S_2^l}\right)}{k_{S_1^l}+k_{S_2^l}}\right| \tag{59}$$

$$\left|\frac{\partial P(g_4)}{\partial \gamma_{2,S_1^l,S_1^l}}\right| = \left|\frac{\left(k_{S_1^l}-1\right)\left[\left(1-\alpha_{1,S_1^l,S_1^l}\right)\left(\eta_{3,S_1^l,S_1^l}-\gamma_{3,S_1^l,S_1^l}\right)\left(\eta_{4,S_1^l,S_1^l}-\gamma_{4,S_1^l,S_1^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}}\right| \tag{60}$$

$$\left|\frac{\partial P(g_4)}{\partial \gamma_{2,S_2^l,S_2^l}}\right| = \left|\frac{\left(k_{S_2^l}-1\right)\left[\left(1-\alpha_{1,S_2^l,S_2^l}\right)\left(\eta_{3,S_2^l,S_2^l}-\gamma_{3,S_2^l,S_2^l}\right)\left(\eta_{4,S_2^l,S_2^l}-\gamma_{4,S_2^l,S_2^l}\right)\right]}{k_{S_1^l}+k_{S_2^l}}\right| \tag{61}$$

$$\left|\frac{\partial P(g_4)}{\partial \gamma_{4,S_2^l,S_1^l}}\right| = \left|\frac{\left[\left(1-\alpha_{1,S_2^l,S_1^l}\right)\left(\eta_{2,S_2^l,S_1^l}-\gamma_{2,S_2^l,S_1^l}\right)+1-\eta_{2,S_2^l,S_1^l}\right]\left(\eta_{3,S_2^l,S_1^l}-\gamma_{3,S_2^l,S_1^l}\right)+1-\eta_{3,S_2^l,S_1^l}}{k_{S_1^l}+k_{S_2^l}}\right| \tag{62}$$

$$\left|\frac{\partial P(g_4)}{\partial \gamma_{4,S_1^l,S_2^l}}\right| = \left|\frac{\left[\left(1-\alpha_{1,S_1^l,S_2^l}\right)\left(\eta_{2,S_1^l,S_2^l}-\gamma_{2,S_1^l,S_2^l}\right)+1-\eta_{2,S_1^l,S_2^l}\right]\left(\eta_{3,S_1^l,S_2^l}-\gamma_{3,S_1^l,S_2^l}\right)+1-\eta_{3,S_1^l,S_2^l}}{k_{S_1^l}+k_{S_2^l}}\right| \tag{63}$$

$$\left|\frac{\partial P\left(g_{\mathrm{bp}}\right)}{\partial \gamma_{3,S_2^l,S_1^l}}\right| = \left|\frac{\left[\left(1-\alpha_{1,S_2^l,S_1^l}\right)\left(\eta_{2,S_2^l,S_1^l}-\gamma_{2,S_2^l,S_1^l}\right)+1-\eta_{2,S_2^l,S_1^l}\right]\left(\eta_{4,S_2^l,S_1^l}-\gamma_{4,S_2^l,S_1^l}\right)}{k_{S_1^l}+k_{S_2^l}}\right| \tag{64}$$

$$\left|\frac{\partial P\left(g_{\mathrm{bp}}\right)}{\partial \gamma_{3,S_1^l,S_2^l}}\right| = \left|\frac{\left[\left(1-\alpha_{1,S_1^l,S_2^l}\right)\left(\eta_{2,S_1^l,S_2^l}-\gamma_{2,S_1^l,S_2^l}\right)+1-\eta_{2,S_1^l,S_2^l}\right]\left(\eta_{4,S_1^l,S_2^l}-\gamma_{4,S_1^l,S_2^l}\right)}{k_{S_1^l}+k_{S_2^l}}\right| \tag{65}$$

$$\left|\frac{\partial P(g_4)}{\partial \gamma_{2,S_2^l,S_1^l}}\right| = \left|\frac{\left(1-\alpha_{1,S_2^l,S_1^l}\right)\left(\eta_{3,S_2^l,S_1^l}-\gamma_{3,S_2^l,S_1^l}\right)\left(\eta_{4,S_2^l,S_1^l}-\gamma_{4,S_2^l,S_1^l}\right)}{k_{S_1^l}+k_{S_2^l}}\right| \tag{66}$$

$$\left|\frac{\partial P(g_4)}{\partial \gamma_{2,S_1^l,S_2^l}}\right| = \left|\frac{\left(1-\alpha_{1,S_1^l,S_2^l}\right)\left(\eta_{3,S_1^l,S_2^l}-\gamma_{3,S_1^l,S_2^l}\right)\left(\eta_{4,S_1^l,S_2^l}-\gamma_{4,S_1^l,S_2^l}\right)}{k_{S_1^l}+k_{S_2^l}}\right| \tag{67}$$

---

product S_m^l, stage i is the quality bottleneck with respect to repair probabilities with defective incoming parts.

Definition 5: For $i \neq j$, if $|[\partial P(g_n)/\partial \gamma_{i,S_{m-1}^l,S_m^l}]| > |[\partial P(g_n)/\partial \gamma_{j,S_{m-1}^l,S_m^l}]|$, then under sequence S^l, $\gamma_{i,S_{m-1}^l,S_m^l}$ is the EQB $-\ \gamma$ for transition m, which means when transiting from product S_{m-1}^l to S_m^l, stage i is the quality bottleneck with respect to failure probabilities with good incoming parts.

Definition 6: For $i \neq j$, if $|[\partial P(g_n)/\partial \mu_{i,S_{m-1}^l,S_m^l}]| > |[\partial P(g_n)/\partial \mu_{j,S_{m-1}^l,S_m^l}]|$, then under sequence S^l, μ_{i,S_{m-1}^l,S_m^l} is the EQB $-\ \mu$ for transition m, which means when transiting from product S_{m-1}^l to S_m^l, stage i is the quality bottleneck with respect to repair probabilities with good incoming parts.

Definition 7: For $i \neq j$, if $|[\partial P(g_n)/\partial \eta_{i,S_{m-1}^l,S_m^l}]| > |[\partial P(g_n)/\partial \eta_{j,S_{m-1}^l,S_m^l}]|$, then under sequence S^l, η_{i,S_{m-1}^l,S_m^l} is the EQB $-\ \eta$ for transition m, which means when transiting from product S_{m-1}^l to S_m^l, stage i is the quality bottleneck with respect to failure probabilities with defective incoming parts.

Definition 8: For $i \neq j$, if $|[\partial P(g_n)/\partial \theta_{i,S_{m-1}^l,S_m^l}]| > |[\partial P(g_n)/\partial \theta_{j,S_{m-1}^l,S_m^l}]|$, then under sequence S^l, $\theta_{i,S_{m-1}^l,S_m^l}$ is EQB $-\ \theta$ for transition m, which means when transiting from product S_{m-1}^l to S_m^l, stage i is the quality bottleneck with respect to repair probabilities with defective incoming parts.

For instance, according to (45), the final quality $P(g_4)$ for a two-product-four-stage system is (55), as shown at the top of the previous page.

From (55), we have (56)–(67), as shown at the top of this page.

By comparing (56), (58), and (60), we find the IQB $-\ \gamma$ for product S_1^l which is the stage

Fig. 5. Four types of products manufactured by the multistage system.

with $\max\{[\partial P(g_4)/\partial\gamma_{4,S_1^l,S_1^l}]\ [\partial P(g_4)/\partial\gamma_{3,S_1^l,S_1^l}]\ [\partial P(g_4)/\partial\gamma_{2,S_1^l,S_1^l}]\}$. Comparing (58), (60), and (62), we obtain the IQB $-\ \gamma$ for product S_2^l which is the stage with $\max\{[\partial P(g_4)/\partial\gamma_{4,S_2^l,S_2^l}]\ [\partial P(g_4)/\partial\gamma_{3,S_2^l,S_2^l}]\ [\partial P(g_4)/\partial\gamma_{2,S_2^l,S_2^l}]\}$. We define (56)–(61) as IQB indicators for quality failure probabilities for good incoming parts. Similarly, (62), (64), and (66) indicate the EQB $-\ \gamma$ for product S_1^l and (63), (65), and (67) indicate the EQB $-\ \gamma$ for product S_2^l. We define (62)–(67) as EQB indicators for quality failure probabilities for defective incoming parts.

Two conclusions can be made from (55)–(67).

Conclusion 6: IQB indicators are related to the transition probabilities of both upstream and downstream stages as well as the batch size of the product.

Conclusion 7: EQB indicators are only related to the transition probabilities of both upstream and downstream stages.

VII. Case Study

To validate the applicability of the proposed method, a case study has been conducted. To ensure the confidentiality of the data, all the parameters introduced below have been modified and only used for illustration.

A. Manufacturing System Description

The model is applied to a four-product-five-stage manufacturing system to evaluate the quality performance of the system. The four types of products (valve shells) that need to be manufactured in this system are shown in Fig. 5. There are five dependent stages (OP 10, 20, 30, 40, and 50) that the four types of products will go through. The relationships among these five stages and the quality propagation are discussed in detail in [32].

B. Results and Discussions

1) Model Prediction Error: Taking product 1 for an example. According to historical data, the internal transition probabilities of product 1 are calculated as $\alpha_1 = 0.05$, $\beta_1 = 0.94$, $\gamma_i = [0.05, 0.1, 0.07, 0.04]$, $\mu_i = [0.92, 0.87, 0.91, 0.95]$, $\eta_i = [0.52, 0.55, 0.43, 0.57]$, and $\theta_i = [0.45, 0.45, 0.55, 0.45]$. Based on the developed model, the quality changes of product 1 along the five-stage system can be estimated. The result is shown in Fig. 6.

The probability of producing a good product estimated from the model is 89.46%. The actual final quality based on historical data is 89.71%, and the prediction error is 0.25%. The result demonstrates the effectiveness and practicability of the proposed model. For all of these four products, the average prediction error is about 0.24%.

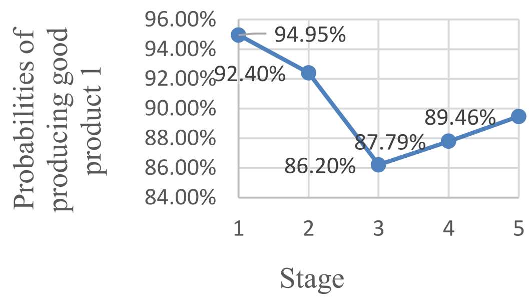

Fig. 6. Estimated probabilities of producing good product 1 through the five-stage system.

2) Production Sequence Analysis: According to the historical data, the failure probabilities and repair probabilities approximately follow Bernoulli distribution, which allows us to study the system under Bernoulli case. According to *Conclusion 3*, to obtain the best sequence, we need to know the various external transition probabilities shown in Table I. They are obtained through the following process.

For a certain part j processed by M_i, either good or defective, there exist four possible statuses.

1) Both parts $(j-1)$ and j are good; thus, M_i maintains a good state g_i.
2) Part $(j-1)$ is good but part j is defective; thus, M_i transits from g_i to d_i.
3) The part $(j-1)$ is defective but part j is good; thus, M_i transits from d_i to g_i.
4) Both parts $(j-1)$ and j are defective; thus, M_i maintains a defective state d_i.

The transition probabilities are estimated from historical data by calculating the proportions of the status: proportion of status 2) represents α, γ, and η while proportion of status 3) refers to β, μ, and θ. If part $(j-1)$ and part j belong to the same type of product, then the estimated probabilities represent internal transition probabilities. Otherwise, if part $(j-1)$ and part j belong to different types of products, then the estimated probabilities represent external transition probabilities.

According to (38), the sequence with the largest value for four-product-five-stage systems is the best sequence.

TABLE I
EXTERNAL TRANSITION PROBABILITIES OF THE SYSTEM

	(1,2)	(1,3)	(1,4)	(2,1)	(2,3)	(2,4)
α_1	0.0667	0.0792	0.0989	0.0717	0.0598	0.0952
γ_2	0.0643	0.0852	0.0972	0.0916	0.0837	0.0779
γ_3	0.1259	0.1208	0.1390	0.1010	0.1409	0.1230
γ_4	0.1068	0.1120	0.0840	0.0958	0.0888	0.1083
γ_5	0.0478	0.0829	0.0623	0.0458	0.0589	0.0635
η_2	0.6085	0.5775	0.5711	0.5482	0.6118	0.6887
η_3	0.6860	0.6491	0.6948	0.6111	0.6087	0.7403
η_4	0.5710	0.5818	0.5981	0.4468	0.5520	0.6047
η_5	0.7205	0.6849	0.7113	0.7278	0.6925	0.6930
	(3,1)	(3,2)	(3,4)	(4,1)	(4,2)	(4,3)
α_1	0.0898	0.0830	0.0574	0.0998	0.0770	0.0507
γ_2	0.0925	0.0951	0.0667	0.0532	0.0780	0.0956
γ_3	0.1336	0.1233	0.1479	0.1237	0.1323	0.1287
γ_4	0.0831	0.1053	0.1062	0.0833	0.0908	0.0978
γ_5	0.0704	0.0497	0.0774	0.0583	0.0470	0.0573
η_2	0.6162	0.6464	0.6798	0.6470	0.6616	0.5658
η_3	0.5924	0.7131	0.5557	0.6662	0.6641	0.6110
η_4	0.4841	0.5294	0.5832	0.6016	0.5680	0.5067
η_5	0.7486	0.7256	0.6620	0.6348	0.6905	0.7580

The optimization model can be expressed as

$$\begin{aligned}\max\ & \sum_{i=1}^{4}\sum_{j=1}^{4} y_{i,j}\Big[\big(1-\gamma_{5,i,j}-\gamma_{4,i,j}(\eta_{5,i,j}-\gamma_{5,i,j})\\ &\quad -\gamma_{3,i,j}(\eta_{4,i,j}-\gamma_{4,i,j})(\eta_{5,i,j}-\gamma_{5,i,j})\\ &\quad +(1-\alpha_{1,i,j})\gamma_{2,i,j}(\eta_{3,i,j}-\gamma_{3,i,j})(\eta_{4,i,j}-\gamma_{4,i,j})\\ &\quad \times(\eta_{5,i,j}-\gamma_{5,i,j})-\alpha_{1,i,j}\eta_{2,i,j}(\eta_{3,i,j}-\gamma_{3,i,j})\\ &\quad \times(\eta_{4,i,j}-\gamma_{4,i,j})(\eta_{5,i,j}-\gamma_{5,i,j})\Big]\\ \text{s.t.}\ & \sum_{i=1}^{4} y_{i,j}=1,\ (i\neq j)\\ & \sum_{j=1}^{4} y_{i,j}=1,\ (i\neq j)\\ & \varphi_i-\varphi_j+Ny_{i,j}\le N-1,\ (i\neq j;\ i=2,3,4;\ j=2,3,4)\end{aligned}$$

where all $y_{i,j}=0$ or 1 and all $\varphi_i>0$; when $y_{i,j}=1$ product i is processed right before product j in the batch cycle, otherwise, $y_{i,j}=0$, $\varphi_i>0$. By solving the model, we obtain $y_{1,4}=1$, $y_{2,1}=1$, $y_{3,2}=1$, $y_{4,3}=1$, and other $y_{i,j}=0$. This result indicates that product 1 is processed before product 4, and product 2 is processed before product 1 while product 3 is processed before product 2, and product 4 is processed before product 3. Then in this case, the best sequence for the four-product-two-stage system would be $1\to4\to3\to2$. Under this sequence, the corresponding probabilities are $\alpha_{1,1,4}$, $\alpha_{1,4,3}$, $\alpha_{1,3,2}$, $\alpha_{1,2,1}$, $\gamma_{i,1,4}$, $\gamma_{i,4,3}$, $\gamma_{i,3,2}$, $\gamma_{i,2,1}$, $\eta_{i,1,4}$, $\eta_{i,4,3}$, $\eta_{i,3,2}$, and $\eta_{i,2,1}$ $(i=2,3,4,5)$.

Based on the above analysis, we can obtain the EQB for each kind of parameters. Similar to (62), (64), and (66), we can obtain the derivatives of $P(g_5)$ with respect to each external transition probabilities. As the comparison of derivatives is independent of the batch size, thus, it can be neglected. Therefore as for $\gamma_{i,1,4}$ the quality failure probability with good incoming parts when transiting from products 1–4, the related derivatives under sequence $1\to4\to3\to2$ are calculated as $[\partial P(g_5)/\partial\gamma_{5,1,4}]=0.8034$, $[\partial P(g_5)/\partial\gamma_{4,1,4}]=0.7695$, $[\partial P(g_5)/\partial\gamma_{3,1,4}]=0.2856$, and $[\partial P(g_5)/\partial\gamma_{2,1,4}]=0.1671$. Since $[\partial P(g_5)/\partial\gamma_{5,1,4}]$ has the largest value, the fifth stage OP50 is the EQB for the quality failure probability with good incoming parts when transiting from products 1–4 in this case. The improvement of $\gamma_{5,1,4}$ can bring the largest benefit to the final quality.

3) Bottleneck Analysis: Also, take product 1 for an example. The quality has the biggest decrease in the third stage (Fig. 6). Taking the third stage into consideration, we can do the sensitivity analysis to find out which transition probability $P(g_3)$is most sensitive. In this case, γ_3, μ_3, η_3, and θ_3 are increased or decreased by given percentages and the sensitivities with respect to the four parameters at 10% are $S_{\gamma_3}=9.43\%$, $S_{\mu_3}=12.13\%$, $S_{\eta_3}=4.21\%$, and $S_{\theta_5}=0.61\%$, respectively, which indicates that the quality $P(g_3)$ is most sensitive to the quality failure probability for good incoming parts μ_3. Based on the model and Definitions 1–4, we can find the IQB for this system. For the repair probability with good incoming parts μ_i, the relationships between μ_i and $P(g_4)$ are shown in Fig. 7.

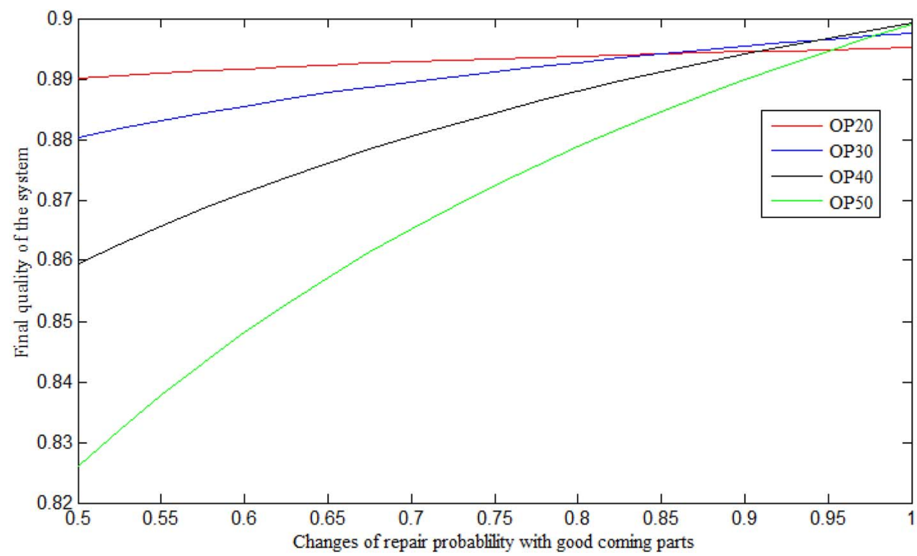

Fig. 7. Relationships between μ_i and $P(g_4)$.

From Fig. 7, we can see that changes of μ_5 have the greatest impact on the final quality $P(g_5)$. The derivatives of $P(g_5)$ also show the same result. The corresponding derivatives are $[\partial P(g_5)/\partial \mu_2] = 0.0065$, $[\partial P(g_5)/\partial \mu_3] = 0.0255$, $[\partial P(g_5)/\partial \mu_4] = 0.0570$, and $[\partial P(g_5)/\partial \mu_5] = 0.0938$. As $[\partial P(g_5)/\partial \mu_5]$ is the largest among these derivatives, the fifth stage OP50 is the IQB for repair probability with good incoming parts μ_i in this case. This means changes of μ_5 can lead to the largest improvement to the final quality.

VIII. Conclusion

M^{3}Ss have been widely applied in industry. It is very important to develop a proper method to evaluate the quality performance of M^{3}Ss. This paper is devoted to modeling and analyzing steady-state M^{3}Ss for quality improvement and filling the gap between FMSs and quality propagation among multiple stages. Two novel Markov models for multiproduct-two-stage systems and multiproduct-multistage systems are developed for quality improvement. The models take both product flexibility and quality propagation into consideration. Several important quality properties including production strategy, product sequence, and quality bottleneck identification are analyzed and a few practical conclusions are noted to provide some insights on quality improvement. Finally, a case study on valve shells is conducted to illustrated the effectiveness of the proposed models. The average prediction error of the models is about 0.24%.

Based on the proposed models in this paper, future work can be conducted as follows.

1) The Markov models can be extended to the one charactering the transient-state quality performance. A Markov model is desirable to be developed for both steady-state and transient-state quality performance.
2) Some other properties in M^{3}Ss for quality improvement can be investigated, such as monotonicity, settling time, and quality loss.
3) The future work also can be directed to modeling and analysis of serial-parallel M^{3}Ss.

Appendix
Proofs

A. Proof of Conclusion 3

According to (39), when there are n types of products, under BP, the difference between the probabilities of producing good products with sequence S^l and S^m is (A1), as shown at the top of the next page, where

$$\sum_{i=1}^{n} k_{S_i^l} = \sum_{i=1}^{n} k_{S_i^m} \tag{A2}$$

$$\begin{aligned}\sum_{i=1}^{n}\left(k_{S_i^l}-1\right)&\left(1-\alpha_{S_i^l,S_i^l}\eta_{S_i^l,S_i^l}+\alpha_{S_i^l,S_i^l}\gamma_{S_i^l,S_i^l}-\gamma_{S_i^l,S_i^l}\right)\\ =\sum_{i=1}^{n}\left(k_{S_i^m}-1\right)&\left(1-\alpha_{S_i^m,S_i^m}\eta_{S_i^m,S_i^m}\right.\\ &\left.+\alpha_{S_i^m,S_i^m}\gamma_{S_i^m,S_i^m}-\gamma_{S_i^m,S_i^m}\right).\end{aligned} \tag{A3}$$

Thus, A(1) can be simplified as (A4), as shown at the top of the next page.

On the other hand, under SP, the difference between probabilities of producing good product with sequence S^l and S^m is (A5), as shown at the top of the next page.

Comparing A(1) with A(5), it can be seen that they have the same signature, which means the best sequences under BP and SP are consistent with each other: $P(g_{\text{bp}}^l) - P(g_{\text{bp}}^m) \Leftrightarrow P(g_{ss}^l) - P(g_{ss}^m)$.

B. Proof of Conclusion 4

Under Bernoulli case, assume that $\delta_{i,j}^1 = 1-\alpha_{1,S_i^l,S_i^l}-\beta_{1,S_i^l,S_i^l}$, $\delta_{i,j}^2 = 1-\gamma_{2,S_i^l,S_i^l}-\mu_{2,S_i^l,S_i^l}$, $\delta_{i,j}^3 = 1-\eta_{2,S_i^l,S_i^l}-\theta_{2,S_i^l,S_i^l}$ and the

$$
\begin{aligned}
&P\left(g_{\mathrm{bp}}^{l}\right)-P\left(g_{\mathrm{bp}}^{m}\right)\\
&=\frac{\sum_{i=1}^{n}\left(k_{S_i^l}-1\right)\left(1-\alpha_{1,S_i^l,S_i^l}\eta_{2,S_i^l,S_i^l}+\alpha_{1,S_i^l,S_i^l}\gamma_{2,S_i^l,S_i^l}-\gamma_{2,S_i^l,S_i^l}\right)}{\sum_{i=1}^{n}k_{S_i^l}}\\
&+\frac{\sum_{i=2}^{n}\left(1-\alpha_{1,S_{i-1}^l,S_i^l}\eta_{2,S_{i-1}^l,S_i^l}+\alpha_{1,S_{i-1}^l,S_i^l}\gamma_{2,S_{i-1}^l,S_i^l}-\gamma_{2,S_{i-1}^l,S_i^l}\right)+\left(1-\alpha_{1,S_n^l,S_1^l}\eta_{2,S_n^l,S_1^l}+\alpha_{1,S_n^l,S_1^l}\gamma_{2,S_n^l,S_1^l}-\gamma_{2,S_n^l,S_1^l}\right)}{\sum_{i=1}^{n}k_{S_i^l}}\\
&-\frac{\sum_{i=1}^{n}\left(k_{S_i^m}-1\right)\left(1-\alpha_{1,S_i^m,S_i^m}\eta_{2,S_i^m,S_i^m}+\alpha_{1,S_i^m,S_i^m}\gamma_{2,S_i^m,S_i^m}-\gamma_{2,S_i^m,S_i^m}\right)}{\sum_{i=1}^{n}k_{S_i^m}}\\
&-\frac{\sum_{i=2}^{n}\left(1-\alpha_{1,S_{i-1}^m,S_i^m}\eta_{2,S_{i-1}^m,S_i^m}+\alpha_{1,S_{i-1}^m,S_i^m}\gamma_{2,S_{i-1}^m,S_i^m}-\gamma_{2,S_{i-1}^m,S_i^m}\right)+\left(1-\alpha_{1,S_n^m,S_1^m}\eta_{2,S_n^m,S_1^m}+\alpha_{1,S_n^m,S_1^m}\gamma_{2,S_n^m,S_1^m}-\gamma_{2,S_n^m,S_1^m}\right)}{\sum_{i=1}^{n}k_{S_i^m}}
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
&P\left(g_{\mathrm{bp}}^{l}\right)-P\left(g_{\mathrm{bp}}^{m}\right)\\
&=\frac{\sum_{i=2}^{n}\left(1-\alpha_{1,S_{i-1}^l,S_i^l}\eta_{2,S_{i-1}^l,S_i^l}+\alpha_{1,S_{i-1}^l,S_i^l}\gamma_{2,S_{i-1}^l,S_i^l}-\gamma_{2,S_{i-1}^l,S_i^l}\right)+\left(1-\alpha_{1,S_n^l,S_1^l}\eta_{1,S_n^l,S_1^l}+\alpha_{1,S_n^l,S_1^l}\gamma_{2,S_n^l,S_1^l}-\gamma_{2,S_n^l,S_1^l}\right)}{\sum_{i=1}^{n}k_{S_i^l}}\\
&-\frac{\sum_{i=2}^{n}\left(1-\alpha_{1,S_{i-1}^m,S_i^m}\eta_{1,S_{i-1}^m,S_i^m}+\alpha_{1,S_{i-1}^m,S_i^m}\gamma_{1,S_{i-1}^m,S_i^m}-\gamma_{1,S_{i-1}^m,S_i^m}\right)+\left(1-\alpha_{1,S_n^m,S_1^m}\eta_{2,S_n^m,S_1^m}+\alpha_{2,S_n^m,S_1^m}\gamma_{2,S_n^m,S_1^m}-\gamma_{1,S_n^m,S_1^m}\right)}{\sum_{i=1}^{n}k_{S_i^l}}
\end{aligned}
\tag{A4}
$$

$$
\begin{aligned}
&P\left(g_{ss}^{l}\right)-P\left(g_{ss}^{m}\right)\\
&=\frac{\sum_{i=2}^{n}\left(1-\alpha_{1,S_{i-1}^l,S_i^l}\eta_{2,S_{i-1}^l,S_i^l}+\alpha_{1,S_{i-1}^l,S_i^l}\gamma_{2,S_{i-1}^l,S_i^l}-\gamma_{2,S_{i-1}^l,S_i^l}\right)+\left(1-\alpha_{1,S_n^l,S_1^l}\eta_{2,S_n^l,S_1^l}+\alpha_{1,S_n^l,S_1^l}\gamma_{2,S_n^l,S_1^l}-\gamma_{2,S_n^l,S_1^l}\right)}{n}\\
&-\frac{\sum_{i=2}^{n}\left(1-\alpha_{1,S_{i-1}^m,S_i^m}\eta_{2,S_{i-1}^m,S_i^m}+\alpha_{1,S_{i-1}^m,S_i^m}\gamma_{2,S_{i-1}^m,S_i^m}-\gamma_{2,S_{i-1}^m,S_i^m}\right)+\left(1-\alpha_{1,S_n^m,S_1^m}\eta_{2,S_n^m,S_1^m}+\alpha_{1,S_n^m,S_1^m}\gamma_{2,S_n^m,S_1^m}-\gamma_{2,S_n^m,S_1^m}\right)}{n}
\end{aligned}
\tag{A5}
$$

matrix of state transition probabilities will be

$$
C=\begin{bmatrix}
0 & C_{1,2} & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & \ddots & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & \cdots & C_{i,k_{S_i^l}} & 0 & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & C_{i+1,1} & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & C_{i+1,2} & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \ddots & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & C_{n,k_{S_n^l}}\\
C_{1,1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\tag{A6}
$$

where (A7), as shown at the top of the next page.

$C_{i,j}$ can be written as the summation of $C_{i,j}^1$ and $C_{i,j}^2$, and (A8) and (A9), as shown at the top of the next page.

According to (26)

$$B=[B_1 \quad B_2 \quad \cdots \quad B_n] \tag{A10}$$

$$B_1=B_2=\cdots=B_n=\begin{bmatrix}0 & 0 & 0 & \frac{1}{K}\end{bmatrix}. \tag{A11}$$

Then for a general case (A12), as shown at the top of the next page.

In the Bernoulli case where $C_{i,j}^2=0$, (A13), as shown at the top of the page following the next page

Defining function F as the summation of the first and second column of the fourth row in a 4×4 matrix, we have

$$P\left(g^l\right)=\frac{1}{K}\sum_{i=1}^{n}\sum_{j=1}^{k_{S_i^l}}F\left[\left(C_{i,j}^1+C_{i,j}^2\right)^{-1}\right] \tag{A14}$$

$$\chi^l=\frac{1}{K}\sum_{i=1}^{n}\sum_{j=1}^{k_{S_i^l}}F\left[\left(C_{i,j}^1\right)^{-1}\right] \tag{A15}$$

$$
\begin{aligned}
&P\left(g^l\right)-\frac{1}{1+\delta_{\max}}\chi^l\\
&=\frac{\sum_{i=1}^{n}\sum_{j=1}^{k_{S_i^l}}\left\{(1+\delta_{\max})F\left[\left(C_{i,j}^1+C_{i,j}^2\right)^{-1}\right]-F\left[\left(C_{i,j}^1\right)^{-1}\right]\right\}}{K(1+\delta_{\max})}.
\end{aligned}
\tag{A16}
$$

Calculate the block matrixes $(C_{i,j}^1+C_{i,j}^2)^{-1}$ and $(C_{i,j}^1)^{-1}$, we obtain that when $\delta_{\max}\ll 1(1+\delta_{\max})F[(C_{i,j}^1+C_{i,j}^2)^{-1}]-F[(C_{i,j}^1)^{-1}]\geq 0$ which means $P(g^l)-[1/(1+\delta_{\max})]\chi^l\geq 0$. Similarly, $P(g^l)-[1/(1-\delta_{\max})]\chi^l\leq 0$ can be proved.

$$C_{i,j} = \begin{bmatrix} \left(1-\alpha_{1,S_i^l,S_i^l}\right)\left(1-\gamma_{2,S_i^l,S_i^l}\right)-1 & \alpha_{1,S_i^l,S_i^l}\left(1-\gamma_{2,S_i^l,S_i^l}\right) & \left(1-\alpha_{1,S_i^l,S_i^l}\right)\gamma_{S_i^l,S_i^l} & 1 \\ \beta_{S_i^l,S_i^l}\left(1-\eta_{S_i^l,S_i^l}\right) & \left(1-\beta_{S_i^l,S_i^l}\right)\left(1-\eta_{S_i^l,S_i^l}\right)-1 & \beta_{S_i^l,S_i^l}\eta_{S_i^l,S_i^l} & 1 \\ \left(1-\alpha_{S_i^l,S_i^l}\right)\mu_{S_i^l,S_i^l} & \alpha_{S_i^l,S_i^l}\mu_{S_i^l,S_i^l} & \left(1-\alpha_{S_i^l,S_i^l}\right)\left(1-\mu_{S_i^l,S_i^l}\right)-1 & 1 \\ \beta_{S_i^l,S_i^l}\theta_{S_i^l,S_i^l} & \left(1-\beta_{S_i^l,S_i^l}\right)\theta_{S_i^l,S_i^l} & \beta_{S_i^l,S_i^l}\left(1-\theta_{S_i^l,S_i^l}\right) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(1-\alpha_{S_i^l,S_i^l}\right)\left(1-\gamma_{S_i^l,S_i^l}\right)-1 & \alpha_{S_i^l,S_i^l}\left(1-\gamma_{S_i^l,S_i^l}\right) & \left(1-\alpha_{S_i^l,S_i^l}\right)\gamma_{S_i^l,S_i^l} & 1 \\ \left(1-\alpha_{S_i^l,S_i^l}\right)\left(1-\eta_{S_i^l,S_i^l}\right) & \alpha_{S_i^l,S_i^l}\left(1-\eta_{S_i^l,S_i^l}\right)-1 & \left(1-\alpha_{S_i^l,S_i^l}\right)\eta_{S_i^l,S_i^l} & 1 \\ \left(1-\alpha_{S_i^l,S_i^l}\right)\left(1-\gamma_{S_i^l,S_i^l}\right) & \alpha_{S_i^l,S_i^l}\left(1-\gamma_{S_i^l,S_i^l}\right) & \left(1-\alpha_{S_i^l,S_i^l}\right)\gamma_{S_i^l,S_i^l}-1 & 1 \\ \left(1-\alpha_{S_i^l,S_i^l}\right)\left(1-\eta_{S_i^l,S_i^l}\right) & \alpha_{S_i^l,S_i^l}\left(1-\eta_{S_i^l,S_i^l}\right) & \left(1-\alpha_{S_i^l,S_i^l}\right)\eta_{S_i^l,S_i^l} & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\delta_{i,j}^1\left(1-\eta_{S_i^l,S_i^l}\right) & \delta_{i,j}^1\left(1-\eta_{S_i^l,S_i^l}\right) & -\delta_{i,j}^1\eta_{S_i^l,S_i^l} & 0 \\ -\delta_{i,j}^2\left(1-\alpha_{S_i^l,S_i^l}\right) & -\delta_{i,j}^2\alpha_{S_i^l,S_i^l} & \delta_{i,j}^2\left(1-\alpha_{S_i^l,S_i^l}\right) & 0 \\ \delta_{i,j}^3\left(1-\alpha_{S_i^l,S_i^l}\right)-\delta_{i,j}^1\left(1-\eta_{S_i^l,S_i^l}-\delta_{i,j}^3\right) & -\delta_{i,j}^3\alpha_{S_i^l,S_i^l}+\delta_{i,j}^1\left(1-\eta_{S_i^l,S_i^l}-\delta_{i,j}^3\right) & \delta_{i,j}^3\left(1-\alpha_{S_i^l,S_i^l}-\delta_{i,j}^1\right)-\delta_{i,j}^1\eta_{S_i^l,S_i^l} & 0 \end{bmatrix} \tag{A7}$$

$$C_{i,j}^1 = \begin{bmatrix} \left(1-\alpha_{S_i^l,S_i^l}\right)\left(1-\gamma_{S_i^l,S_i^l}\right)-1 & \alpha_{S_i^l,S_i^l}\left(1-\gamma_{S_i^l,S_i^l}\right) & \left(1-\alpha_{S_i^l,S_i^l}\right)\gamma_{S_i^l,S_i^l} & 1 \\ \left(1-\alpha_{S_i^l,S_i^l}\right)\left(1-\eta_{S_i^l,S_i^l}\right) & \alpha_{S_i^l,S_i^l}\left(1-\eta_{S_i^l,S_i^l}\right)-1 & \left(1-\alpha_{S_i^l,S_i^l}\right)\eta_{S_i^l,S_i^l} & 1 \\ \left(1-\alpha_{S_i^l,S_i^l}\right)\left(1-\gamma_{S_i^l,S_i^l}\right) & \alpha_{S_i^l,S_i^l}\left(1-\gamma_{S_i^l,S_i^l}\right) & \left(1-\alpha_{S_i^l,S_i^l}\right)\gamma_{S_i^l,S_i^l}-1 & 1 \\ \left(1-\alpha_{S_i^l,S_i^l}\right)\left(1-\eta_{S_i^l,S_i^l}\right) & \alpha_{S_i^l,S_i^l}\left(1-\eta_{S_i^l,S_i^l}\right) & \left(1-\alpha_{S_i^l,S_i^l}\right)\eta_{S_i^l,S_i^l} & 1 \end{bmatrix} \tag{A8}$$

$$C_{i,j}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\delta_{i,j}^1\left(1-\eta_{S_i^l,S_i^l}\right) & \delta_{i,j}^1\left(1-\eta_{S_i^l,S_i^l}\right) & -\delta_{i,j}^1\eta_{S_i^l,S_i^l} & 0 \\ -\delta_{i,j}^2\left(1-\alpha_{S_i^l,S_i^l}\right) & -\delta_{i,j}^2\alpha_{S_i^l,S_i^l} & \delta_{i,j}^2\left(1-\alpha_{S_i^l,S_i^l}\right) & 0 \\ \delta_{i,j}^3\left(1-\alpha_{S_i^l,S_i^l}\right)-\delta_{i,j}^1\left(1-\eta_{S_i^l,S_i^l}-\delta_{i,j}^3\right) & -\delta_{i,j}^3\alpha_{S_i^l,S_i^l}+\delta_{i,j}^1\left(1-\eta_{S_i^l,S_i^l}-\delta_{i,j}^3\right) & \delta_{i,j}^3\left(1-\alpha_{S_i^l,S_i^l}-\delta_{i,j}^1\right)-\delta_{i,j}^1\eta_{S_i^l,S_i^l} & 0 \end{bmatrix} \tag{A9}$$

$$X = BC^{-1} = B\left(C^1+C^2\right)^{-1} = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}$$
$$\times \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \left(C_{1,1}^1+C_{1,1}^2\right)^{-1} \\ \left(C_{1,2}^1+C_{1,2}^2\right)^{-1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & \left(C_{i,k_{S_i^l}}^1+C_{i,k_{S_i^l}}^2\right)^{-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \left(C_{i+1,1}^1+C_{i+1,1}^2\right)^{-1} & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \left(C_{i+1,2}^1+C_{i+1,2}^2\right)^{-1} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \left(C_{n,k_{S_n^l}}^1+C_{n,k_{S_n^l}}^2\right)^{-1} & 0 \end{bmatrix} \tag{A12}$$

C. Proof of Conclusion 5

According to (39) and (40), $\chi_{\rm bp}^l$, $\chi_{\rm bp}^m$, $\chi_{\rm sp}^l$, and $\chi_{\rm sp}^m$ can be obtained, (A17)–(A20), as shown at the top of the next page.

Then according to (A2) and (A3)

$$\frac{\chi_{ss}^l-\chi_{ss}^m}{\chi_{ss}^l+\chi_{ss}^m} = \frac{\chi_{\rm bp}^l-\chi_{\rm bp}^m}{\chi_{\rm bp}^l+\chi_{\rm bp}^m}.$$

$$X' = BC^{-1} = B\left(C^1\right)^{-1}$$
$$= \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \left(C^1_{1,1}\right)^{-1} \\ \left(C^1_{1,2}\right)^{-1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & \left(C^1_{i,k_{S_i^l}}\right)^{-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \left(C^1_{i+1,1}\right)^{-1} & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \left(C^1_{i+1,2}\right)^{-1} & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \left(C^1_{n,k_{S_n^l}}\right)^{-1} & 0 \end{bmatrix} \tag{A13}$$

$$\chi^l_{\text{bp}} = P\left(g^l_{\text{bp}}\right) = \frac{\sum_{i=1}^n \left(k_{S_i^l} - 1\right)\left(1 - \alpha_{1,S_i^l,S_i^l}\eta_{2,S_i^l,S_i^l} + \alpha_{1,S_i^l,S_i^l}\gamma_{1,S_i^l,S_i^l} - \gamma_{2,S_i^l,S_i^l}\right)}{\sum_{i=1}^n k_{S_i^l}}$$
$$+ \frac{\sum_{i=2}^n \left(1 - \alpha_{1,S_{i-1}^l,S_i^l}\eta_{1,S_{i-1}^l,S_i^l} + \alpha_{1,S_{i-1}^l,S_i^l}\gamma_{2,S_{i-1}^l,S_i^l} - \gamma_{2,S_{i-1}^l,S_i^l}\right) + \left(1 - \alpha_{1,S_n^l,S_1^l}\eta_{2,S_n^l,S_1^l} + \alpha_{1,S_n^l,S_1^l}\gamma_{2,S_n^l,S_1^l} - \gamma_{2,S_n^l,S_1^l}\right)}{\sum_{i=1}^n k_{S_i^l}} \tag{A17}$$

$$\chi^m_{\text{bp}} = P\left(g^m_{\text{bp}}\right) = \frac{\sum_{i=1}^n \left(k_{S_i^m} - 1\right)\left(1 - \alpha_{1,S_i^m,S_i^m}\eta_{2,S_i^m,S_i^m} + \alpha_{1,S_i^m,S_i^m}\gamma_{2,S_i^m,S_i^m} - \gamma_{2,S_i^m,S_i^m}\right)}{\sum_{i=1}^n k_{S_i^m}}$$
$$+ \frac{\sum_{i=2}^n \left(1 - \alpha_{1,S_{i-1}^m,S_i^m}\eta_{2,S_{i-1}^m,S_i^m} + \alpha_{1,S_{i-1}^m,S_i^m}\gamma_{2,S_{i-1}^m,S_i^m} - \gamma_{2,S_{i-1}^m,S_i^m}\right) + \left(1 - \alpha_{1,S_n^m,S_1^m}\eta_{2,S_n^m,S_1^m} + \alpha_{1,S_n^m,S_1^m}\gamma_{2,S_n^m,S_1^m} - \gamma_{2,S_n^m,S_1^m}\right)}{\sum_{i=1}^n k_{S_i^m}} \tag{A18}$$

$$\chi^l_{\text{sp}} = P\left(g^l_{ss}\right)$$
$$= \frac{\sum_{i=2}^n \left(1 - \alpha_{1,S_{i-1}^l,S_i^l}\eta_{2,S_{i-1}^l,S_i^l} + \alpha_{1,S_{i-1}^l,S_i^l}\gamma_{2,S_{i-1}^l,S_i^l} - \gamma_{2,S_{i-1}^l,S_i^l}\right) + \left(1 - \alpha_{1,S_n^l,S_1^l}\eta_{2,S_n^l,S_1^l} + \alpha_{1,S_n^l,S_1^l}\gamma_{2,S_n^l,S_1^l} - \gamma_{2,S_n^l,S_1^l}\right)}{n} \tag{A19}$$

$$\chi^m_{\text{sp}} = P\left(g^m_{ss}\right)$$
$$= \frac{\sum_{i=2}^n \left(1 - \alpha_{1,S_{i-1}^m,S_i^m}\eta_{2,S_{i-1}^m,S_i^m} + \alpha_{1,S_{i-1}^m,S_i^m}\gamma_{2,S_{i-1}^m,S_i^m} - \gamma_{2,S_{i-1}^m,S_i^m}\right) + \left(1 - \alpha_{1,S_n^m,S_1^m}\eta_{2,S_n^m,S_1^m} + \alpha_{1,S_n^m,S_1^m}\gamma_{2,S_n^m,S_1^m} - \gamma_{2,S_n^m,S_1^m}\right)}{n} \tag{A20}$$

Based on Conclusion 4

$$\frac{1}{1+\delta_{\max}}\chi^l < P\left(g^l\right) < \frac{1}{1-\delta_{\max}}\chi^l$$
$$\frac{\left(\chi^l_{ss} - \chi^m_{ss}\right) - \delta_{\max}\left(\chi^l_{ss} + \chi^m_{ss}\right)}{1-\delta^2_{\max}}$$
$$< P\left(g^l_{ss}\right) - P\left(g^m_{ss}\right) < \frac{\left(\chi^l_{ss} - \chi^m_{ss}\right) + \delta_{\max}\left(\chi^l_{ss} + \chi^m_{ss}\right)}{1-\delta^2_{\max}} \tag{A21}$$

$$\frac{\left(\chi^l_{\text{bp}} - \chi^m_{\text{bp}}\right) - \delta_{\max}\left(\chi^l_{\text{bp}} + \chi^m_{\text{bp}}\right)}{1-\delta^2_{\max}}$$
$$< P\left(g^l_{\text{bp}}\right) - P\left(g^m_{\text{bp}}\right) < \frac{\left(\chi^l_{\text{bp}} - \chi^m_{\text{bp}}\right) + \delta_{\max}\left(\chi^l_{\text{bp}} + \chi^m_{\text{bp}}\right)}{1-\delta^2_{\max}} \tag{A22}$$

when $0 < \delta_{\max} < [(\chi^l_{ss} - \chi^m_{ss})/(\chi^l_{ss} + \chi^m_{ss})] = [(\chi^l_{\text{bp}} - \chi^m_{\text{bp}})/(\chi^l_{\text{bp}} + \chi^m_{\text{bp}})]$, we have $(\chi^l_{ss} - \chi^m_{ss}) - \delta_{\max}(\chi^l_{ss} + \chi^m_{ss}) > 0$ and $(\chi^l_{\text{bp}} - \chi^m_{\text{bp}}) - \delta_{\max}(\chi^l_{\text{bp}} + \chi^m_{\text{bp}}) > 0$, then

$P(g^l_{ss}) - P(g^m_{ss}) > 0$ and $P(g^l_{\text{bp}}) - P(g^m_{\text{bp}}) > 0$, which means $P(g^l_{\text{bp}}) > P(g^m_{\text{bp}}) \Leftrightarrow P(g^l_{ss}) > P(g^m_{ss})$ in the Bernoulli case.

REFERENCES

[1] J. Jin and J. Shi, "State space modeling of sheet metal assembly for dimensional control," *ASME Trans. J. Manuf. Sci. Eng.*, vol. 121, no. 4, pp. 756–762, 1999.

[2] J. Shi, *Stream of Variation Modeling and Analysis for Multistage Manufacturing Processes*. Boca Raton, FL, USA: CRC Press, 2006.

[3] J. Shi and S. Zhou, "Quality control and improvement for multistage systems: A survey," *IIE Trans.*, vol. 41, no. 9, pp. 744–753, 2009.

[4] L. Li and Z. Sun, "Dynamic energy control for energy efficiency improvement of sustainable manufacturing systems using Markov decision process," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 43, no. 5, pp. 1195–1205, Sep. 2013.

[5] R. R. Inman, D. E. Blumenfeld, N. Huang, J. Li, and J. Li, "Survey of recent advances on the interface between production system design and quality," *IIE Trans.*, vol. 45, no. 6, pp. 557–574, 2013.

[6] F. Ju, J. Li, G. Xiao, and J. Arinez, "Quality flow model in automotive paint shops," *Int. J. Prod. Res.*, vol. 51, no. 21, pp. 6470–6483, 2013.

[7] F. Ju, J. Li, G. Xiao, N. Huang, and S. Biller, "A quality flow model in battery manufacturing systems for electric vehicles," *IEEE Trans. Autom. Sci. Eng.*, vol. 11, no. 1, pp. 230–244, Jan. 2014.

[8] J. Wang, J. Li, J. Arinez, S. Biller, and N. Huang, "Quality analysis in flexible manufacturing systems with batch productions," in *Proc. IEEE Int. Conf. Autom. Sci. Eng.*, Washington, DC, USA, 2008, pp. 460–465.

[9] J. Wang, J. Li, J. Arinez, and S. Biller, "Product sequencing with respect to quality in flexible manufacturing systems with batch operations," *IEEE Trans. Autom. Sci. Eng.*, vol. 7, no. 4, pp. 776–790, Oct. 2010.

[10] J. Wang, J. Li, J. Arinez, and S. Biller, "Indicators for quality improvability and bottleneck sequence in flexible manufacturing systems with batch production," *Int. J. Prod. Res.*, vol. 50, no. 22, pp. 6388–6402, 2012.

[11] J. Wang, J. Li, J. Arinez, and S. Biller, "Quality bottleneck transitions in flexible manufacturing systems with batch productions," *IIE Trans.*, vol. 45, no. 2, pp. 190–205, 2013.

[12] Y. Chen, Z. Li, and M. Zhou, "Behaviorally optimal and structurally simple liveness-enforcing supervisors of flexible manufacturing systems," *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans*, vol. 42, no. 3, pp. 615–629, May 2012.

[13] Y. Chen, Z. Li, and A. Al-Ahmari, "Nonpure Petri net supervisors for optimal deadlock control of flexible manufacturing systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 43, no. 2, pp. 252–265, Mar. 2013.

[14] D. Liu, Z. Li, and M. Zhou, "Hybrid liveness-enforcing policy for generalized Petri net models of flexible manufacturing systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 43, no. 1, pp. 85–97, Jan. 2013.

[15] Z. Li, M. Zhou, and N. Wu, "A survey and comparison of Petri net-based deadlock prevention policies for flexible manufacturing systems," *IEEE Trans. Syst., Man, Cybern. C, Appl. Rev.*, vol. 38, no. 2, pp. 173–188, Mar. 2008.

[16] A. K. Sethi and S. P. Sethi, "Flexibility in manufacturing: A survey," *Int. J. Flexible Manuf. Syst.*, vol. 2, no. 4, pp. 289–328, 1990.

[17] R. Beach, A. P. Muhlemann, D. H. R. Price, A. Paterson, and J. A. Sharp, "A review of manufacturing flexibility," *Eur. J. Oper. Res.*, vol. 122, no. 1, pp. 41–57, 2000.

[18] A. Jain, P. K. Jain, F. T. S. Chan, and S. Singh, "A review on manufacturing flexibility," *Int. J. Prod. Res.*, vol. 51, no. 19, pp. 5946–5970, 2013.

[19] J. F. Lawless, R. J. Mackay, and J. A. Robinson, "Analysis of variation transmission in manufacturing processes—Part I," *J. Qual. Technol.*, vol. 31, no. 2, pp. 131–142, 1999.

[20] S. Du and L. Xi, "Fault diagnosis in assembly processes based on engineering-driven rules and PSOSAEN algorithm," *Comput. Ind. Eng.*, vol. 60, no. 1, pp. 77–88, 2011.

[21] N. Jin and S. Zhou, "Data-driven variation source identification of manufacturing processes based on eigenspace comparison," *Naval Res. Logist.*, vol. 53, no. 5, pp. 383–396, 2006.

[22] S. C. Mondal, J. Maiti, and P. K. Ray, "Modelling robustness in serial multi-stage manufacturing processes," *Int. J. Prod. Res.*, vol. 51, no. 21, pp. 6359–6377, 2013.

[23] Y. Ding, D. M. Ceglarek, and J. Shi, "Design evaluation of multi-station assembly processes by using state space approach," *J. Mech. Design*, vol. 124, no. 3, pp. 408–418, 2002.

[24] W. Huang, J. Lin, Z. Kong, and D. Ceglarek, "Stream-of-variation (SOVA) modeling II: A generic 3D variation model for rigid body assembly in multistation assembly processes," *ASME Trans. J. Manuf. Sci. Eng.*, vol. 129, no. 4, pp. 832–842, 2007.

[25] J.-P. Loose, N. Chen, and S. Zhou, "Surrogate modeling of dimensional variation propagation in multistage assembly processes," *IIE Trans.*, vol. 41, no. 10, pp. 893–904, 2009.

[26] Q. Zhou, P. Z. G. Qiang, and S. Zhou, "Surrogate modeling of multistage assembly processes using integrated emulation," *ASME Trans. J. Mech. Design*, vol. 134, no. 1, pp. 1–10, 2012.

[27] D. Djurdjanovic and J. Ni, "Dimensional errors of fixtures, locating and measurement datum features in the stream of variation modeling in machining," *ASME Trans. J. Manuf. Sci. Eng.*, vol. 125, no. 4, pp. 716–730, 2003.

[28] Q. Huang, J. Shi, and J. Yuan, "Part dimensional error and its propagation modeling in multi-operational machining processes," *ASME Trans. J. Manuf. Sci. Eng.*, vol. 125, no. 2, pp. 255–262, 2003.

[29] S. Zhou, Q. Huang, and J. Shi, "State space modeling of dimensional variation propagation in multistage machining process using differential motion vectors," *IEEE Trans. Robot. Autom.*, vol. 19, no. 2, pp. 296–309, Apr. 2003.

[30] S. Zhou, Y. Chen, and J. Shi, "Root cause estimation and statistical testing for quality improvement of multistage manufacturing processes," *IEEE Trans. Autom. Sci. Eng.*, vol. 1, no. 1, pp. 73–83, Jul. 2004.

[31] J. V. Abellan-Nebot, J. Liu, F. R. Subirón, and J. Shi, "State space modeling of variation propagation in multistation machining processes considering machining-induced variations," *ASME Trans. J. Manuf. Sci. Eng.*, vol. 134, no. 2, 2012, Art. no. 021002.

[32] S. Du, X. Yao, D. Huang, and M. Wang, "Three-dimensional variation propagation modeling for multistage turning process of rotary workpieces," *Comput. Ind. Eng.*, vol. 82, pp. 41–53, Apr. 2015.

[33] S. Du, X. Yao, and D. Huang, "Engineering model-based Bayesian monitoring of ramp-up phase of multistage manufacturing process," *Int. J. Prod. Res.*, vol. 53, no. 15, pp. 4594–4613, 2015.

[34] F. Ju, J. Li, G. Xiao, and J. Arinez, "Quality flow model in automotive paint shops," *Int. J. Prod. Res.*, vol. 51, no. 21, pp. 6470–6483, 2013.

[35] F. Ju, J. Li, G. Xiao, N. Huang, and S. Biller, "A quality flow model in battery manufacturing systems for electric vehicles," *IEEE Trans. Autom. Sci. Eng.*, vol. 11, no. 1, pp. 230–244, Jan. 2014.

[36] J. Kim and S. B. Gershwin, "Analysis of long flow lines with quality and operational failure," *IIE Trans.*, vol. 40, no. 3, pp. 284–296, 2008.

[37] J. Li, D. E. Blumenfeld, and S. P. Marin, "Production system design for quality robustness," *IIE Trans.*, vol. 40, no. 3, pp. 162–176, 2008.

[38] M. Colledani and T. Tolio, "Joint design of quality and production control in manufacturing systems," *CIRP J. Manuf. Sci. Technol.*, vol. 4, no. 3, pp. 281–289, 2011.

[39] S. Ioannidis, "Joint production and quality control in production systems with two customer classes and lost sales," *IIE Trans.*, vol. 45, no. 6, pp. 605–616, 2013.

[40] J. Kim and S. B. Gershwin, "Integrated quality and quantity modeling of a production line," *OR Spectr.*, vol. 27, no. 2, pp. 287–314, 2005.

[41] J. Li and N. Huang, "Quality evaluation in flexible manufacturing systems: A Markovian approach," *Math. Problems Eng.*, vol. 2007, May 2007, Art. no. 57128.

[42] C. Zhao and J. Li, "Analysis and improvement of multi-product assembly systems: An application study at a furniture manufacturing plant," *Int. J. Prod. Res.*, vol. 52, no. 21, pp. 6399–6413, 2014.

[43] L. Li and Z. Sun, "Dynamic energy control for energy efficiency improvement of sustainable manufacturing systems using Markov decision process," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 43, no. 5, pp. 1195–1205, Sep. 2013.

[44] S. Du, R. Xu, D. Huang, and X. Yao, "Markov modeling and analysis of multi-stage manufacturing systems with remote quality information feedback," *Comput. Ind. Eng.*, vol. 88, pp. 13–25, Oct. 2015.

[45] Y. Wang and L. Li, "A novel modeling method for both steady-state and transient analyses of serial Bernoulli production system," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 45, no. 1, pp. 97–108, Jan. 2015.

[46] C. Zhao and J. Li, "Analysis and improvement of multiproduct Bernoulli serial lines: Theory and application," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 45, no. 9, pp. 1218–1230, Sep. 2015.

Shichang Du (M'16) received the B.S. and M.S.E. degrees in mechanical engineering from the Hefei University of Technology, Hefei, China, in 2000 and 2003, respectively, and the Ph.D. degree in industrial engineering and management from Shanghai Jiao Tong University, Shanghai, China, in 2008.

He was a Visiting Scholar with the University of Michigan, Ann Arbor, MI, USA, from 2006 to 2007. He is an Associate Professor with Shanghai Jiao Tong University. His current research interests include quality and reliability engineering, quality control with analysis of error flow, and monitoring and diagnosis of manufacturing process.

Rui Xu received the B.S. degree from the School of Mechanical Engineering, Shanghai Jiao Tong University, Shanghai, China, in 2013, where she is currently pursuing master's degree in industrial engineering and management.

Her current research interests include Markov modeling and analysis of multistage manufacturing systems.

Lin Li (M'12) received the B.S. degree in mechanical engineering from Shanghai Jiao Tong University, Shanghai, China, in 2001, and the M.S.E. degrees in mechanical engineering and industrial and operations engineering and the Ph.D. degree in mechanical engineering from the University of Michigan, Ann Arbor, MI, USA, in 2003, 2005, and 2007, respectively.

He is currently an Associate Professor with the Department of Mechanical and Industrial Engineering and the Director of Sustainable Manufacturing Systems Research Laboratory, University of Illinois at Chicago, Chicago, IL, USA. His current research interests include sustainable manufacturing systems, reliability engineering, intelligent maintenance systems, and healthcare systems.