Modeling and Analysis of Multiproduct Multistage Manufacturing System for Quality Improvement

Shichang Du, Member, IEEE, Rui Xu, and Lin Li, Member, IEEE

Abstract—The ability to produce multiple types of products using the same manufacturing system plays an essential role in the success of a manufacturing enterprise. Meanwhile, most complex manufacturing systems include many stages, called multistage manufacturing systems (MMSs). MMS has a cascade property, which means the product quality is not only affected by the current stage, but is also related to the outputs of the upstream stage. Multiproduct MMSs (MPSs) have been widely applied in industry. Thus, this paper is devoted to modeling and analyzing steady-state MPSs for quality improvement. The discrete Markov model for single-product-multistage system is extended to novel Markov models for multistage-two-stage systems and multiproduct-multistage systems by calculating state transition probabilities of six key manufacturing parameters to obtain an acceptable product quality probability. Based on the developed models, product sequence analysis is carried out to obtain the best product sequence under Bernoulli conditions and Bernoulli relaxation conditions, and bottleneck analysis under Bernoulli conditions is conducted to identify the machine and/or parameters whose improvement can lead to the largest quality improvement. The applicability of the proposed models is validated through numerical experiments and a case study using real-world data.

Index Terms—Bottleneck, Markov model, multiple products, multistage manufacturing system (MMS), quality control.

NOMENCLATURE

$M_i$ ith stage in multistage manufacturing system (MMS).

$M'_i$ Stage merged by the first $i$ stages in MMSs.

$k_i$ Batch size of product $j$.

$K$ Total amount of products produced in one batch.

$g_i$ $M_i$ or $M'_i$ is producing a good product.

$d_i$ $M_i$ or $M'_i$ is producing a defective product.

$S_l$ One possible sequence when producing multiple products.

$S_l^i$ $m$th type of product in sequence $S_l$.

$S_m^j$ jth product internal of $S^i_m$.

$s_{i,s_{m,j}}$ $M_i$ or $M'_i$ is in good state $g_i$ when producing product $S^i_{m,j}$.

$d_{i,d_{m,j}}$ $M_i$ or $M'_i$ is in defective state $d_i$ when producing product $S^i_m$.

$\alpha_1$ Probability for $M_1$ to transit from state $g_1$ to state $d_1$.

$\alpha_{1,s_{i,s_{j}}}$ Probability for $M_1$ to transit from state $g_1$ to state $d_1$ when switching from producing $S^i_1$ to producing $S^j_1$.

$\beta_1$ Probability for $M_1$ to transit from state $d_1$ to state $g_1$.

$\beta_{1,s_{i,s_{j}}}$ Probability for $M_1$ to transit from state $d_1$ to state $g_1$ when switching from producing $S^i_1$ to producing $S^j_1$.

$\gamma_i$ In case of good incoming parts, the probability for $M_i$ to transit from state $g_i$ to state $d_i$.

$\gamma_{i,s_{i,s_{j}}}$ In case of good incoming parts, the probability for $M_i$ to transit from state $g_i$ to state $d_i$ when switching from producing $S^i_p$ to producing $S^j_p$.

$\mu_i$ In case of good incoming parts, the probability for $M_i$ to transit from state $d_i$ to state $g_i$.

$\mu_{i,s_{i,s_{j}}}$ In case of good incoming parts, the probability for $M_i$ to transit from state $d_i$ to state $g_i$ when switching from producing $S^i_p$ to producing $S^j_p$.

$\eta_i$ In case of defective incoming parts, the probability for $M_i$ to transit from state $g_i$ to state $d_i$.

$\eta_{i,s_{i,s_{j}}}$ In case of defective incoming parts, the probability for $M_i$ to transit from state $g_i$ to state $d_i$ when switching from producing $S^i_p$ to producing $S^j_p$.

$\theta_i$ In case of defective incoming parts, the probability for $M_i$ to transit from state $d_i$ to state $g_i$.

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S. Du and R. Xu are with the Department of Industrial Engineering and Management, Shanghai Jiao Tong University, Shanghai 200240, China (e-mail: lowbin@sjtu.edu.cn; lucyruixu@126.com).

L. Li is with the Department of Industrial and Manufacturing Engineering, University of Illinois at Chicago, Chicago, IL 60607 USA (e-mail: linli@uic.edu).

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In case of defective incoming parts, the probability for \( M_i \) to transit from state \( d_i \) to state \( g_i \), when switching from producing \( S^l_i \) to producing \( S^l_j \).

- \( A_i \) = Matrix of state transition probabilities for the system with \( i \) stages.
- \( A_{m,j} \) = Matrix of state transition probabilities for the system with \( i \) stages when producing the \( j \)th product internal of \( S^l_i \).
- \( X_i \) = Matrix of steady-state probabilities for the system with \( i \) stages.
- \( X_{m,j} \) = Matrix of steady-state probabilities for the system with \( i \) stages when producing the \( j \)th product internal of \( S^l_i \).
- \( P \) = Probability of the system in one certain steady state.
- \( P(g_i) \) = Probability of the system with \( i \) stages in a good state.
- \( P(g_{bp}) \) = Probability that the system produces good products under batch production (BP) with sequence \( S^l \).
- \( P(g_{bp}) \) = Probability that the system produces good products under single production (SP) with sequence \( S^l \).
- \( S_{r1} \) = Sensitivity of \( P(g_i) \) with respect to \( \gamma_i \).
- \( S_{d1} \) = Sensitivity of \( P(g_i) \) with respect to \( \mu_i \).
- \( S_{p1} \) = Sensitivity of \( P(g_i) \) with respect to \( \eta_i \).
- \( S_{h1} \) = Sensitivity of \( P(g_i) \) with respect to \( \theta_i \).
- \( S_{r1} \) = Sensitivity of \( P(g_i) \) with respect to \( \gamma_1 \).
- \( S_{p1} \) = Sensitivity of \( P(g_i) \) with respect to \( \beta_1 \).
- \( \delta_1 \) = Maximum difference between 1 and \( (\alpha_{1,i},S^l_j) + (\beta_{1,i},S^l_j) \).
- \( \delta_2 \) = Maximum difference between 1 and \( (\gamma_{2,i},S^l_j) + (\mu_{2,i},S^l_j) \).
- \( \delta_3 \) = Maximum difference between 1 and \( (\eta_{2,i},S^l_j) + \theta_{2,i},S^l_j \).

**I. INTRODUCTION**

The ability to produce multiple types of products using the same manufacturing system plays an essential role in the success of a manufacturing enterprise. Product flexibility increases the rapid responsiveness of a system; it takes full advantage of available system resources to produce multiple types of products using the same manufacturing system that deals with internal and external production uncertainties with time. Meanwhile, most complex manufacturing systems involve many stages, called MMSs. As products move through these stages, the variations in product quality are usually introduced and propagated. Multiproduct MMSs (M3Ss) are becoming more and more popular and necessary and have been widely applied in automobile vehicles, engine, aerospace, electronics, and appliance industry.

In M3Ss, the final product quality is dependent on not only product flexibility but also product quality propagation. For example, flexible fixtures play an important role to enable flexibility of the whole manufacturing system, and they are programmable in order to hold and clamp different types of products in a same manufacturing system. The fixtures locating accuracy determine the product quality. When the product is changed from one type to another one, the fixtures need to adapt themselves to the desired corresponding positions. Since there exist relocation errors of the fixtures, their conditions could be changed from “good” to “bad.” Assume there are products A and B machined in two-stage system (see Fig. 1). If the fixture is in good condition for product A and the subsequent product is also product A at stage 1, then a good quality product would be produced. However, if the subsequent product is changed from products A to B, then the fixture needs to readjust its position (there exists a transition probability from good position to bad position) and either good quality or defective products at stage 1 may be produced. The defective product B may be produced at stage 1. Thus, the product quality at stage 1 is dependent on not only the quality of the previous product but also the product type, namely, the product flexibility can affect the quality. Meanwhile, if the characteristic of defective product B (caused by product flexibility) produced at stage 1 is used as the locating datum to machine other characteristics of product B at stage 2, then the quality variation of product B would propagate from stages 1 to 2. Flexibility can affect the quality propagation. Therefore, both product flexibility and quality propagation could significantly affect the final product quality in M3Ss. However, there has been no analytical model to investigate and improve product quality considering both product flexibility and product quality propagation in M3Ss.

Modeling and analysis of MMSs for quality propagation has received intensive investigation. One of the most popular analytical models used for quality improvement is state space model [1]. After that, a great deal of extended research on quality propagation based on state space model has been conducted. Detailed descriptions of existing research on the state space model are provided in a monograph [2] and a survey [3]. In another research line of analytical methods, a Markov model has been widely used as an analytical tool to investigate the interactions between the manufacturing system and product quality [4]. The related literatures are reviewed in [5]. However, in spite of above efforts, the current Markov models are explored only for single-product-multistage systems without product flexibility [6], [7] or multiproduct-single-stage system without quality propagation [8]–[11]. To the best knowledge of the authors, there lacks an efficient method to model and analyze M3Ss which features both quality propagation and product flexibility.

The main contribution of this paper is to propose novel Markov models for steady-state M3Ss for quality improvement, which take both product flexibility and product quality propagation into consideration.

The remainder of this paper is organized as follows. Section II reviews the related literature. Section III introduces
M3Ss and formulates the problem. Two Markov models for multiproduct-two-stage system and multiproduct-multistage system are developed, respectively, in Section IV. Product sequence and quality improvement analysis are carried out in Sections V and VI. A case study is presented to illustrate the proposed models and their applicability in Section VII. Finally, the conclusion is given in Section VIII.

II. Literature Review

Product flexibility has been widely recognized as a critical component to achieve a competitive advantage in the market place. Flexible manufacturing systems (FMSs) have been widely designed and applied [12]–[14]. Detailed descriptions of existing research on manufacturing flexibility analysis methods and applications were reviewed in [15]–[18].

Modeling and analysis of product quality propagation in MMSs have attracted substantial research attention in recent decades. Data-driven methods focus on investigating patterns from massive historical quality datasets to model the relationships between product quality and manufacturing systems [19]–[22].

Unlike data-driven models, analytical models employ offline analysis of MMSs based on fundamental physical laws. The state space model is one of the most popular analytical models used for quality propagation analysis [1]. It is further investigated in 3-D assembly systems [23]–[26] and machining systems [27]–[33]. Detailed descriptions of the existing research on state space model are provided in a monograph [2] and a survey [3]. However, analysis of complex MMSs using the state space model based on physical laws is often intractable [3], and such analysis either relies on complicated kinematics models of manufacturing systems, or is only applicable to deal with dimensional errors and the application area is limited [34], [35].

In another research concentration, Markov models have been widely used as analytical tools to investigate the product quality. The related literature and empirical evidences show that manufacturing system design has a significant impact on product quality [5]. Modeling and analysis of manufacturing systems using Markov models for product quality improvement have received more and more research attention.

Some Markov models are developed to analyze product quality propagation in single-product-multistage systems. Markov-chain-based quality propagation models are developed to evaluate quality propagation in automotive paint shops [6] and battery assembly process [7]. The analytical methods using Markov models are proposed to evaluate three cases of long manufacturing lines [36]. The impact of manufacturing system design on product quality is investigated through a case study at an automotive paint shop [37]. Also, analytical methods are proposed for the joint design of quality and manufacturing parameters [38] and investigate joint production and quality control in manufacturing systems with random demand [39].

Some other Markov models are derived to analyze the quality of multiproduct-single-stage systems. The early Markov model is developed to study how production system design, quality and productivity are inter-related [40]. Then a Markov model to evaluate quality performance of multiproduct-single-stage systems is presented in [41]. Based on Markov chain processes, some analytical methods are developed to evaluate quality performance in multiproduct manufacturing systems with BP in [8].

Several research studies on product sequencing and bottlenecks using Markov models in multiproduct-single-stage systems to improve product quality have been conducted. The impact of product sequencing and batch policies on product quality is investigated and some insights to achieve better quality are presented [9]. Quality bottleneck analysis is carried out based on data from a factory floor [10], [11]. An arrow-based bottleneck identification method is presented in [6]. A Markov model is applied to characterize a furniture assembly system in [42]. A Markov model is explored to control dynamic energy for energy efficiency improvement of sustainable manufacturing systems [43].

In spite of the above efforts, M3Ss with quality propagation and product flexibility still lacks an in-depth study, and there is no efficient Markov model to analyze these systems for quality improvement. The goal of this paper is to contribute to this end.

III. Model Assumptions and Problem Formulation

Consider an n-product-r-stage system; the following assumptions define the processing stages, product types, and their interactions in the Markov models.

1) The manufacturing system consists of r stages which can produce n different types of products. The corresponding batch size for product j is \(k_j(1 \leq j \leq n)\). The total amount of products produced in one batch is \(K = \sum_{i=1}^{n} k_i\).

2) We only study the working or production period of the system. Machine breakdowns are not considered.

3) Define the stage \(M_i(i = 1, 2, \ldots, r)\) in a good state \(g_i\) or a defective state \(d_i\) if it is producing a product with good quality or defective quality at time t. The quality of incoming product is characterized by good state \(g_i\) or defective state \(d_i\) with probabilities \(P(g_i)\) and \(P(d_i)\), respectively.

4) There exist quality degradation and quality correction in the system. The quality might get worse or better after a certain stage. The quality of the incoming parts for \(M_i(i \geq 2)\) at time \((t + 1)\) depends on the state of \(M_{i-1}\) at time t. The states \(g_{i-1}\) and \(d_{i-1}\) for \(M_{i-1}\) at time t means good and defective parts coming for \(M_i\) at time \((t + 1)\), respectively.

5) When \(M_i\) is in good state \(g_i\), it has probability \(\alpha_i\) to transit to defective state \(d_i\) and probability \((1 - \alpha_i)\) to good state \(g_i\). When \(M_i\) is in defective state \(d_i\), it has probability \(\beta_i\) to transit to good state \(g_i\) and probability \((1 - \beta_i)\) to defective state \(d_i\).

6) With good incoming parts, when \(M_i(i \geq 2)\) is in good state \(g_i\), it has probability \(\gamma_i\) to transit to defective state \(d_i\) and probability \((1 - \gamma_i)\) to good state \(g_i\). When \(M_i\) is in
Fig. 2. Framework of the proposed method.

For the system producing $n$ different types of products, there are $(n-1)!$ possible production sequences. For a certain sequence $S^l = \{S^l_1, S^l_2, \ldots, S^l_m\}$, $S^l_m$ denotes the $m$th type of product in this sequence, where $m \in \{1, 2, \ldots, n\}$.

9) When $M_i$ or $M_{i+1}$ is processing $S^l_{m,j}$ at time $t$, it may maintain processing the same type of product $S^l_{m,j+1}$ or switch to processing another type of product $S^l_{m+1,1}$ at $(t+1)$. Then the corresponding transition probabilities for $M_i$ and $M_{i+1}$ are denoted as $\alpha_{1,S^l_{p,q}}, \beta_{1,S^l_{p,q}}$ and $\gamma_{2,S^l_{p,q}}, \mu_{2,S^l_{p,q}}, \eta_{2,S^l_{p,q}}$, and $\theta_{2,S^l_{p,q}}$ ($p, r = 1, 2, \ldots, n$). When $p = r$, these probabilities imply the internal transition probabilities of the same type of product. When $p \neq r$, they indicate the external transition probabilities between different types of products.

We refer to $\alpha_1, \gamma_1, \eta_1 (i \geq 2)$ as quality failure probabilities and $(i \geq 2)\beta_1, \mu_1, \theta_1$ as quality repair probabilities. Similar to throughput analysis and in accordance with some quality analysis work based on Markov model [6], [7], [41], we assume that all these transition probabilities are constant. Actually in real manufacturing systems, machines have stable

production periods during which the state transitions can be seen as stable.

The problem addressed is then formulated as follows.

**Problem:** Under the above assumptions 1)–9), develop a proper method to evaluate the steady-state quality performance of $M^3$Ss as a function of system parameters, investigate the sequence properties, perform sensitivity analysis and identify quality bottlenecks.

The solutions to the problem are given in Sections IV–VI. The framework of the proposed method is illustrated in Fig. 2. Based on the Markov model for single-product-two-stage systems in [44], a Markov model for multiproduct-two-stage systems is developed. Then this model is extended to multiproduct-multistage systems. Product sequence analysis and quality improvement analysis are conducted based on the developed models.

IV. MARKOV MODEL FOR $M^3$Ss


Based on the assumptions 1)–9) and the work of [44], for a two-stage system producing a given type of product $S^l_{m,j}$, the transition probabilities and the matrix of state transition probabilities are presented (1), as shown at the top of the next page.

The matrix of steady state probabilities is denoted as

$$X_2 = [P(g_1g_2) \quad P(d_1g_2) \quad P(g_1d_2) \quad P(d_1d_2)].$$

(2)

Based on the Markov model, we have

$$X_2A_2 = X_2$$

(3)

$$P(g_1g_2) + P(d_1g_2) + P(g_1d_2) + P(d_1d_2) = 1$$

(4)

and

$$P(g_2) = P(g_1g_2) + P(d_1g_2).$$

(5)

Equation (5) represents the probability that the system is producing a good product. This probability can be seen as an indicator to evaluate the quality performance of the system.

The model for single-product-two-stage systems can be extended to single-product-$r$-stage systems ($r \geq 3$) by applying the iteration method in [44] which has been validated.
by extensive numerical experiments. The general iterative procedure is illustrated in Fig. 3 and the main steps are presented as follows.

1) Merge $M_1$ and $M_2$ to one stage $M'_2$, and gain the quality of the new two-stage system $M'_2 - M_3$ based on the model for one-product-two-stage system.

2) Merge $M'_2$ and $M_3$ to one stage $M'_3$, and gain the quality of the new two-stage system $M'_3 - M_4$. Continue this iterative process until the first $(r-1)$ stages are merged to one stage $M'_{r-1}$ and gain the quality of the final two-stage system $M'_{r-1} - M_r$.

During the iterative process of single-product-$r$-stage systems, any two-stage system $M'_i - M_{i+1}$ has six basic parameters. They are $Y_{i+1} S_{m,n}, S_{m,n}', \beta_i S_{m,n}, S_{m,n}', \eta_{i+1} S_{m,n}, S_{m,n}', \alpha_i S_{m,n}, S_{m,n}'$, and $\theta_i S_{m,n}, S_{m,n}'$ describing the characteristics of $M_{i+1}$ and $\alpha_i S_{m,n}', S_{m,n}'$, $\beta_i S_{m,n}', S_{m,n}'$ describing the characteristics of the merged machine $M'_i$. $\alpha_i S_{m,n}, S_{m,n}'$, $\beta_i S_{m,n}, S_{m,n}'$ can be calculated by

$$a'_i S_{m,n}, S_{m,n}' = \frac{P(g_{i-1} g_i) Y_{i+1} S_{m,n}, S_{m,n}' + P(d_{i-1} g_i) \eta_{i+1} S_{m,n}, S_{m,n}'}{P(g_{i-1} g_i) + P(d_{i-1} g_i)}$$

$$\beta'_i S_{m,n}, S_{m,n}' = \frac{P(g_{i-1} d_i) \beta_i S_{m,n}, S_{m,n}' + P(d_{i-1} d_i) \eta_{i+1} S_{m,n}, S_{m,n}'}{P(g_{i-1} d_i) + P(d_{i-1} d_i)}.$$  

The matrix of state transition probabilities \( \mathbf{A}_2 \), as shown at the top of the next page, \( \mathbf{A}_2 \) =

\[
\begin{bmatrix}
(1 - \alpha_{1 S_{m,n}, S_{m,n}'})(1 - \gamma_{2 S_{m,n}, S_{m,n}'} ) & \beta_{1 S_{m,n}, S_{m,n}'}(1 - \eta_{2 S_{m,n}, S_{m,n}'} ) & 0 & 0 \\
(1 - \alpha_{1 S_{m,n}, S_{m,n}'})(1 - \gamma_{2 S_{m,n}, S_{m,n}'} ) & \beta_{1 S_{m,n}, S_{m,n}'}(1 - \eta_{2 S_{m,n}, S_{m,n}'} ) & 0 & 0 \\
\rho_{1 S_{m,n}, S_{m,n}'} \alpha_{2 S_{m,n}, S_{m,n}'} & \rho_{1 S_{m,n}, S_{m,n}'} \beta_{2 S_{m,n}, S_{m,n}'} & 0 & 0 \\
\rho_{1 S_{m,n}, S_{m,n}'} \alpha_{2 S_{m,n}, S_{m,n}'} & \rho_{1 S_{m,n}, S_{m,n}'} \beta_{2 S_{m,n}, S_{m,n}'} & 0 & 0 \\
(1 - \alpha_{1 S_{m,n}, S_{m,n}'})(1 - \gamma_{2 S_{m,n}, S_{m,n}'} ) & \beta_{1 S_{m,n}, S_{m,n}'}(1 - \eta_{2 S_{m,n}, S_{m,n}'} ) & 0 & 0 \\
(1 - \alpha_{1 S_{m,n}, S_{m,n}'})(1 - \gamma_{2 S_{m,n}, S_{m,n}'} ) & \beta_{1 S_{m,n}, S_{m,n}'}(1 - \eta_{2 S_{m,n}, S_{m,n}'} ) & 0 & 0 \\
\beta_{1 S_{m,n}, S_{m,n}'} \alpha_{2 S_{m,n}, S_{m,n}'} & \beta_{1 S_{m,n}, S_{m,n}'} \beta_{2 S_{m,n}, S_{m,n}'} & 0 & 0 \\
\beta_{1 S_{m,n}, S_{m,n}'} \alpha_{2 S_{m,n}, S_{m,n}'} & \beta_{1 S_{m,n}, S_{m,n}'} \beta_{2 S_{m,n}, S_{m,n}'} & 0 & 0 \\
\end{bmatrix}
\]

The matrix of steady state probability \( \mathbf{A}_2 \) =

\[
\begin{bmatrix}
\gamma_{2 S_{m-1,n}, S_{m-1,n}'} & \mu_{2 S_{m-1,n}, S_{m-1,n}'} & \eta_{2 S_{m-1,n}, S_{m-1,n}'} & \theta_{2 S_{m-1,n}, S_{m-1,n}'} \\
\gamma_{2 S_{m-1,n}, S_{m-1,n}'} & \mu_{2 S_{m-1,n}, S_{m-1,n}'} & \eta_{2 S_{m-1,n}, S_{m-1,n}'} & \theta_{2 S_{m-1,n}, S_{m-1,n}'} \\
\gamma_{2 S_{m-1,n}, S_{m-1,n}'} & \mu_{2 S_{m-1,n}, S_{m-1,n}'} & \eta_{2 S_{m-1,n}, S_{m-1,n}'} & \theta_{2 S_{m-1,n}, S_{m-1,n}'} \\
\gamma_{2 S_{m-1,n}, S_{m-1,n}'} & \mu_{2 S_{m-1,n}, S_{m-1,n}'} & \eta_{2 S_{m-1,n}, S_{m-1,n}'} & \theta_{2 S_{m-1,n}, S_{m-1,n}'} \\
\gamma_{2 S_{m-1,n}, S_{m-1,n}'} & \mu_{2 S_{m-1,n}, S_{m-1,n}'} & \eta_{2 S_{m-1,n}, S_{m-1,n}'} & \theta_{2 S_{m-1,n}, S_{m-1,n}'} \\
\gamma_{2 S_{m-1,n}, S_{m-1,n}'} & \mu_{2 S_{m-1,n}, S_{m-1,n}'} & \eta_{2 S_{m-1,n}, S_{m-1,n}'} & \theta_{2 S_{m-1,n}, S_{m-1,n}'} \\
\gamma_{2 S_{m-1,n}, S_{m-1,n}'} & \mu_{2 S_{m-1,n}, S_{m-1,n}'} & \eta_{2 S_{m-1,n}, S_{m-1,n}'} & \theta_{2 S_{m-1,n}, S_{m-1,n}'} \\
\gamma_{2 S_{m-1,n}, S_{m-1,n}'} & \mu_{2 S_{m-1,n}, S_{m-1,n}'} & \eta_{2 S_{m-1,n}, S_{m-1,n}'} & \theta_{2 S_{m-1,n}, S_{m-1,n}'} \\
\end{bmatrix}
\]

The sum of all the steady state probabilities equals 1

\[
\sum_{m=1}^{n} \sum_{j=1}^{k} P(g_{2 S_{m,j}, d_{2 S_{m,j}}}) + \sum_{m=1}^{n} \sum_{j=1}^{k} P(d_{2 S_{m,j}, d_{2 S_{m,j}}}) = 1
\]

and

\[
\mathbf{A}_2 \mathbf{X} = \mathbf{X}
\]
\[
A_{i+1} = \begin{bmatrix}
(1 - \alpha'_1 s_{j,m} s_{n,m}) (1 - \gamma_2 s_{j,m} s_{n,m}) & \alpha'_1 s_{j,m} s_{n,m} & (1 - \alpha_1 s_{j,m} s_{n,m}) \gamma_2 s_{j,m} s_{n,m} & \alpha'_1 s_{j,m} s_{n,m} \\
(1 - \alpha'_1 s_{j,m} s_{n,m}) & (1 - \alpha'_1 s_{j,m} s_{n,m}) & (1 - \alpha_1 s_{j,m} s_{n,m}) & \alpha'_1 s_{j,m} s_{n,m} \\
(1 - \alpha'_1 s_{j,m} s_{n,m}) & (1 - \alpha'_1 s_{j,m} s_{n,m}) & \alpha'_1 s_{j,m} s_{n,m} & (1 - \alpha_1 s_{j,m} s_{n,m}) \\
(1 - \alpha'_1 s_{j,m} s_{n,m}) & (1 - \alpha'_1 s_{j,m} s_{n,m}) & \alpha'_1 s_{j,m} s_{n,m} & (1 - \alpha_1 s_{j,m} s_{n,m})
\end{bmatrix}
\]

(8)

\[
X_{m,1} = \begin{bmatrix}
P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) P(d_{1,s_{j,m+1}} s_{2,s_{m,n}}) P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) \\
P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) P(d_{1,s_{j,m+1}} s_{2,s_{m,n}}) P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) \\
P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) P(d_{1,s_{j,m+1}} s_{2,s_{m,n}}) P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) P(g_{1,s_{j,m+1}} s_{2,s_{m,n}})
\end{bmatrix}
\]

(13)

\[
A_{i,1} = \begin{bmatrix}
(1 - \alpha_1 s_{j,m} s_{n,m}) (1 - \gamma_2 s_{j,m} s_{n,m}) & \alpha_1 s_{j,m} s_{n,m} & (1 - \alpha_1 s_{j,m} s_{n,m}) \gamma_2 s_{j,m} s_{n,m} & \alpha_1 s_{j,m} s_{n,m} \\
(1 - \alpha_1 s_{j,m} s_{n,m}) & (1 - \alpha_1 s_{j,m} s_{n,m}) & (1 - \alpha_1 s_{j,m} s_{n,m}) & \alpha_1 s_{j,m} s_{n,m} \\
(1 - \alpha_1 s_{j,m} s_{n,m}) & (1 - \alpha_1 s_{j,m} s_{n,m}) & \alpha_1 s_{j,m} s_{n,m} & (1 - \alpha_1 s_{j,m} s_{n,m}) \\
(1 - \alpha_1 s_{j,m} s_{n,m}) & (1 - \alpha_1 s_{j,m} s_{n,m}) & \alpha_1 s_{j,m} s_{n,m} & (1 - \alpha_1 s_{j,m} s_{n,m})
\end{bmatrix}
\]

(14)

By summing up the elements in \(X_{i,j}\), we find that

\[
P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) + P(d_{1,s_{j,m+1}} s_{2,s_{m,n}}) + P(g_{1,s_{j,m+1}} s_{2,s_{m,n}})
\]

which leads to

\[
P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) + P(d_{1,s_{j,m+1}} s_{2,s_{m,n}}) + P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) + P(g_{1,s_{j,m+1}} s_{2,s_{m,n}}) = \frac{1}{K}.
\]

(26)

The final quality can be obtained from (23–26)

\[
P(g^i) = \sum_{m=1}^{n} \sum_{j=1}^{k} P(g_{2,s_{m,n}}).
\]

(27)
If we ignore quality propagation in the system, namely, \( \eta_2.s_m.s_n = \gamma_2.s_m.s_n, \eta_2.s_{m-1}.s_m = \gamma_2.s_{m-1}.s_m, \) and \( \eta_2.s_1.s_1 = \gamma_2.s_1.s_1, \) then the probability of producing a product with good quality could be (28), as shown at the top of this page, which is consistent with conclusion (26) made in [9].

### C. Proposed Markov Model for Multiproduct-Multistage Systems

Based on the model for single-product-multistage systems that focuses on quality propagation and the model for multiproduct-two-stage systems that focuses on product flexibility, we derive the general model for M^3Ss. The corresponding probabilities are: \( \alpha'_i.s_m.s_m, \alpha''_i.s_m.s_m, \alpha'_i.s_{m-1}.s_m, \beta''_i.s_m.s_m, \beta'_i.s_{m-1}.s_m, \gamma_i.s_m.s_n, \gamma_{i,s_{m-1}}.s_m, \eta_i.s_m.s_n, \eta_{i,s_{m-1}}.s_m, \mu_i.s_{m-1}.s_m, \mu_{i,s_{m-1}}.s_m, \theta_i.s_{m+1}.s_m, \) where \( m = 1, 2, \ldots, r - 1. \)

Especially when \( m = n, \) it means a batch cycle has been finished and the system will enter another production cycle. Under this situation, \( \alpha'_i.s_n.s_n, \) for example, will be denoted as \( \alpha'_i.s_n.s_n. \)

Similar to the matrices of transition probabilities that have been stated in Section IV-B, the transition probability matrices for the products in the first, middle, and last positions are (35)–(37), as shown at the top of the next page.

According to the definition of a Markov chain, we can get the probability of producing a product with good quality for M^3Ss. Especially when the probabilities satisfy Bernoulli distribution, we have \( \alpha_{1,i,j} + \beta_{1,i,j} = 1, \gamma_{i,j} + \mu_2,i,j = 1, \eta_{1,i,j} + \theta_{2,i,j} = 1(i = 1, 2, \ldots, n, j = 1, 2, \ldots, n) \) [8], [40], [45], then the expression of the final quality for a n-product-r-stage system with sequence \( S^j \) can be obtained (38), as shown at the top of the next page.

### V. PRODUCT SEQUENCE ANALYSIS

As indicated in the models for multiproduct-two-stage systems and multiproduct-multistage systems, different sequences involve different transition probabilities, which means the final quality might differ from each other under different sequences. Therefore, an appropriate sequence of multiple products could improve the final product quality. Furthermore, different sequence strategies, BP as \( a, \ldots, a, b, \ldots, b, c, \cdots, c, \ldots, a, \cdots, \) or SP as \( a, b, c, \cdots, a, b, c, \cdots \) also have impact on the quality.

#### A. Bernoulli Case

In order to simplify the analysis and make conclusions more explicit, we first focus on the quality properties regarding the sequence of multiproduct-two-stage systems under the Bernoulli case. The Bernoulli case is often employed in
\[
A_{g_{1}} = \begin{bmatrix}
(1 - \alpha_{r, s_{1}, s_{1}^{*}}) & (1 - \gamma_{r+1, s_{1+1}, s_{1}^{*}}) & \beta_{r, s_{1}, s_{1}^{*}} & (1 - \mu_{1+s_{1}, s_{1}^{*}}) & (1 - \nu_{1+s_{1}, s_{1}^{*}}) \\
\alpha_{r, s_{1}, s_{1}^{*}} & (1 - \gamma_{r+1, s_{1+1}, s_{1}^{*}}) & \beta_{r, s_{1}, s_{1}^{*}} & (1 - \mu_{1+s_{1}, s_{1}^{*}}) & (1 - \nu_{1+s_{1}, s_{1}^{*}}) \\
\alpha_{r, s_{1}, s_{1}^{*}} & \gamma_{r+1, s_{1+1}, s_{1}^{*}} & \beta_{r, s_{1}, s_{1}^{*}} & (1 - \mu_{1+s_{1}, s_{1}^{*}}) & (1 - \nu_{1+s_{1}, s_{1}^{*}}) \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
P(g_{bp}) = \sum_{i=1}^{n} k_{g_{1}}^{i} - 1 \left[ (1 - \alpha_{1, s_{1}, s_{1}^{*}}) \gamma_{2, s_{1}, s_{1}^{*}} \gamma_{r+1, s_{1+1}, s_{1}^{*}} \gamma_{3, s_{1}, s_{1}^{*}} \cdots \gamma_{3, s_{1}, s_{1}^{*}} \prod_{i=3}^{r} (\eta_{i, s_{1}, s_{1}^{*}} - \gamma_{r, s_{1}, s_{1}^{*}}) \right]
\]

\[
P(g_{sp}) = \sum_{i=1}^{n} (k_{g_{1}}^{i} - 1) \left[ (1 - \alpha_{1, s_{1}, s_{1}^{*}}) \gamma_{2, s_{1}, s_{1}^{*}} \gamma_{r+1, s_{1+1}, s_{1}^{*}} \gamma_{3, s_{1}, s_{1}^{*}} \cdots \gamma_{3, s_{1}, s_{1}^{*}} \prod_{i=3}^{r} (\eta_{i, s_{1}, s_{1}^{*}} - \gamma_{r, s_{1}, s_{1}^{*}}) \right]
\]

quality analysis and assumes that the system quality follows a Bernoulli distribution: \(\alpha_{1, i, j} + \beta_{1, i, j} = 1, \gamma_{1, i, j} + \mu_{2, i, j} = 1,\) and \(\eta_{1, i, j} + \nu_{2, i, j} = 1 (i = 1, 2, \ldots, n, j = 1, 2, \ldots, n).\) Accordingly, we obtain the probability that the system produces good products under BP \(P(g_{bp})\) and SP \(P(g_{sp})\), respectively, as shown at the top of this page.
Here we first take a three-product-two-stage system as an example to study the quality properties of product sequence and then extend these properties to more general cases. Assume that three different products \(a, b, c\) are produced. There exist two different sequences: one is \(S^1: a \rightarrow b \rightarrow c\), the other is \(S^2: a \rightarrow c \rightarrow b\). The corresponding probabilities are

\[
P_{bp}(g_{S^1}) = \frac{(k_a - 1)[1 - \gamma_{2,a,a} + \alpha_{1,a,a}(\gamma_{2,a,a} - \eta_{2,a,a})]}{k_a + k_b + k_c} + \frac{(k_b - 1)[1 - \gamma_{2,b,b} + \alpha_{1,b,b}(\gamma_{2,b,b} - \eta_{2,b,b})]}{k_a + k_b + k_c} + \frac{(k_c - 1)[1 - \gamma_{2,c,c} + \alpha_{1,c,c}(\gamma_{2,c,c} - \eta_{2,c,c})]}{k_a + k_b + k_c} + \frac{1 - \gamma_{2,a,a} + \alpha_{1,a,a}(\gamma_{2,a,a} - \eta_{2,a,a})}{k_a + k_b + k_c} + \frac{1 - \gamma_{2,b,b} + \alpha_{1,b,b}(\gamma_{2,b,b} - \eta_{2,b,b})}{k_a + k_b + k_c}
\]

\[
P_{bp}(g_{S^2}) = \frac{(k_a - 1)[1 - \gamma_{2,a,a} + \alpha_{1,a,a}(\gamma_{2,a,a} - \eta_{2,a,a})]}{k_a + k_b + k_c} + \frac{(k_b - 1)[1 - \gamma_{2,b,b} + \alpha_{1,b,b}(\gamma_{2,b,b} - \eta_{2,b,b})]}{k_a + k_b + k_c} + \frac{(k_c - 1)[1 - \gamma_{2,c,c} + \alpha_{1,c,c}(\gamma_{2,c,c} - \eta_{2,c,c})]}{k_a + k_b + k_c} + \frac{1 - \gamma_{2,a,a} + \alpha_{1,a,a}(\gamma_{2,a,a} - \eta_{2,a,a})}{k_a + k_b + k_c} + \frac{1 - \gamma_{2,b,b} + \alpha_{1,b,b}(\gamma_{2,b,b} - \eta_{2,b,b})}{k_a + k_b + k_c}
\]

\[
P_{sp}(g_{S^1}) = \frac{1 - \gamma_{2,c,a} + \alpha_{1,c,a}(\gamma_{2,c,a} - \eta_{2,c,a})}{3} + \frac{1 - \gamma_{2,a,b} + \alpha_{1,a,b}(\gamma_{2,a,b} - \eta_{2,a,b})}{3}
\]

\[
P_{sp}(g_{S^2}) = \frac{1 - \gamma_{2,c,a} + \alpha_{1,c,a}(\gamma_{2,c,a} - \eta_{2,c,a})}{3} + \frac{1 - \gamma_{2,a,b} + \alpha_{1,a,b}(\gamma_{2,a,b} - \eta_{2,a,b})}{3}
\]
B. Bernoulli Relaxation Case

Although the Bernoulli case is very similar to the real manufacturing conditions, it still seems strict to some extent. Thus, we slightly relax the Bernoulli case and extend the model to more general cases in practice. Under the Bernoulli relaxation case, the summation of failure probability and repair probability does not have to equal 1 but just to be close to 1. In other words

$$
\delta_1 = \max \left\{ 1 - \alpha_{i, i'}, j, j' - \beta_{i, i'} j, j' \right\},
$$
$$
\delta_2 = \max \left\{ 1 - \gamma_{i, i'}, j - \mu_{i, i'} j \right\},
$$
$$
\delta_3 = \max \left\{ 1 - \eta_{i, i'}, j - \theta_{i, i'} j \right\},
$$
$$
\delta_{\max} = \max \{ \delta_1, \delta_2, \delta_3 \}
$$

where $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$, and $0 \leq \delta_1, \delta_2, \delta_3, \delta_{\max} < 1$.

**Conclusion 4:** The probability of producing a good product for M3Ss in a certain interval is decided by $\delta_{\max}$ and

$$
\frac{1}{1 + \delta_{\max}} \chi^l \leq P(g^l) \leq \frac{1}{1 - \delta_{\max}} \chi^l \quad (45)
$$

where for BP

$$
\chi^l = \chi^l_{bp} = P \left( g^l_{bp} \right) = \sum_{i=1}^{n} \left( k_{i} - 1 \right) \left( 1 - \alpha_{i, i', j, j'} + \alpha_{i, i', j, j'} \gamma_{i, i', j, j'} - \gamma_{i, i', j, j'} \right)
$$
$$
+ \sum_{i=2}^{n} \left( 1 - \alpha_{i, i', i, j, j'} + \alpha_{i, i', i, j, j'} \gamma_{i, i', i, j, j'} - \gamma_{i, i', i, j, j'} \right)
$$
$$
+ \sum_{i=1}^{n} k_{i} \left( 1 - \alpha_{i, i', j, j'} + \alpha_{i, i', j, j'} \gamma_{i, i', j, j'} - \gamma_{i, i', j, j'} \right)
$$

and for SP

$$
\chi^l = \chi^l_{sp} = P \left( g^l_{sp} \right) = \sum_{i=1}^{n} \left( 1 - \alpha_{i, i', i, j, j'} + \alpha_{i, i', i, j, j'} \gamma_{i, i', i, j, j'} - \gamma_{i, i', i, j, j'} \right)
$$
$$
+ \sum_{i=2}^{n} \left( 1 - \alpha_{i, i', i, j, j'} + \alpha_{i, i', i, j, j'} \gamma_{i, i', i, j, j'} - \gamma_{i, i', i, j, j'} \right)
$$
$$
+ \sum_{i=1}^{n} k_{i} \left( 1 - \alpha_{i, i', i, j, j'} + \alpha_{i, i', i, j, j'} \gamma_{i, i', i, j, j'} - \gamma_{i, i', i, j, j'} \right). \quad (47)
$$

Proof of Conclusion 4 can be found in the Appendix.

**Conclusion 5:** Under Bernoulli relaxation case, when $0 \leq \delta_{\max} < \left[ (\chi^l_{bp} - \chi^m_{bp})/(\chi^l_{sp} + \chi^m_{sp}) \right] = \left[ (\chi^l_{bp} - \chi^m_{bp})/(\chi^l_{bp} + \chi^m_{bp}) \right]$, we still have $P(g^l_{bp}) > P(g^l_{sp}) \Leftrightarrow P(g^m_{bp}) > P(g^m_{sp})$. Proof of Conclusion 5 can be found in the Appendix.

Actually not only when $0 \leq \delta_{\max} < (\chi^l_{sp} - \chi^m_{sp})/(\chi^l_{sp} + \chi^m_{sp})$ but also under most Bernoulli relaxation cases, the conclusion $P(g^l_{bp}) > P(g^m_{bp}) \Leftrightarrow P(g^l_{sp}) > P(g^m_{sp})$ holds. In order to verify it, extensive numerical experiments have been carried out. We assume that $\delta_{\max} \leq 0.2$ and $\alpha \in [0, 0.2]$, $\beta \in [0.8, 1]$, $\gamma \in [0, 0.2]$, $\mu \in [0.8, 1]$, $\eta \in [0, 1]$, and $\theta \in [0, 1]$. Corresponding transition probabilities under certain sequences are randomly generated within their intervals and the results of the final quality are estimated by applying the model for multiproduct-two-stage systems. When there are more than three types of products, alternative sequences can be chosen to improve quality. Here we increase the number of product types from 3 to 6 and the number of possible sequences increases from 2 to 120. For the given number of product types, make comparisons between the results of final quality under different sequences and each comparison is based on 1000 times of numerical experiments. The results show that the probability that $P(g^l_{bp}) > P(g^m_{bp}) \Leftrightarrow P(g^l_{sp}) > P(g^m_{sp})$ holds are around 97% when the number of product types ranges from 3 to 6.

When there are three types of products, the probability that the conclusion $P(g^l_{bp}) > P(g^m_{bp}) \Leftrightarrow P(g^l_{sp}) > P(g^m_{sp})$ holds is about 91%. But when the types of products increase to 4–6, the probabilities are all between 96% and 97% (Fig. 4).

So it is reasonable to conclude that in practice, the best sequences under BP and SP in M3Ss are consistent with each other.

VI. QUALITY IMPROVEMENT ANALYSIS

A. Sensitivity Analysis

The product quality for a multistage system depends on the quality failure probabilities $\gamma_{i}, \eta_{i}$ and quality repair probabilities $\mu_{i}, \theta_{i}$. Changes of these parameters can lead to the improvement of quality $P(g_{i})$. It is necessary to find out which parameter could bring about the largest quality improvement. Sensitivity analysis of $P(g_{i})$ with respect to $\gamma_{i}, \eta_{i}, \mu_{i}, \theta_{i}$ can help figure out this question.

For the sensitivity analysis, we change only one parameter and while the others remain unchanged. Accordingly, the changed parameters and probabilities are $\gamma_{i}^{\prime}, \mu_{i}^{\prime}, \eta_{i}^{\prime}, \theta_{i}^{\prime}$, and $P_{\gamma_{i}}, P_{\mu_{i}}(g_{i}), P_{\eta_{i}}(g_{i}), P_{\theta_{i}}(g_{i})$, respectively. Then the sensitivity of $P(g_{i})$ with respect to $\gamma_{i}, \mu_{i}, \eta_{i}, \theta_{i}$ could be written as

$$
S_{\gamma_{i}} = \frac{|P_{\gamma_{i}}(g_{i}) - P(g_{i})|}{P(g_{i})} \left/ \gamma_{i} \right| \quad (48)
$$
$$
S_{\mu_{i}} = \frac{|P_{\mu_{i}}(g_{i}) - P(g_{i})|}{P(g_{i})} \left/ \mu_{i} \right| \quad (49)
$$
$$
S_{\eta_{i}} = \frac{|P_{\eta_{i}}(g_{i}) - P(g_{i})|}{P(g_{i})} \left/ \eta_{i} \right| \quad (50)
$$
$$
S_{\theta_{i}} = \frac{|P_{\theta_{i}}(g_{i}) - P(g_{i})|}{P(g_{i})} \left/ \theta_{i} \right|. \quad (51)
$$
\[
P(g_1) = \left( k_{s_1} - 1 \right) \left[ 1 - \gamma_{4, s_1, s_1'} - \gamma_{3, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
- \left( k_{s_1} - 1 \right) \left[ \left( 1 - \alpha_{1, s_1, s_1'} \right) \eta_{3, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
- \left( k_{s_1} - 1 \right) \left[ \alpha_{1, s_1, s_1'} \eta_{2, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
+ \left( k_{s_1} - 1 \right) \left[ 1 - \gamma_{4, s_1, s_1'} - \gamma_{3, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
- \left( k_{s_1} - 1 \right) \left[ \left( 1 - \alpha_{1, s_1, s_1'} \right) \eta_{2, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
- \left( k_{s_1} - 1 \right) \left[ \alpha_{1, s_1, s_1'} \eta_{2, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
+ \left( k_{s_1} - 1 \right) \left[ 1 - \gamma_{4, s_1, s_1'} - \gamma_{3, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
- \left( k_{s_1} - 1 \right) \left[ \left( 1 - \alpha_{1, s_1, s_1'} \right) \eta_{2, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
- \left( k_{s_1} - 1 \right) \left[ \alpha_{1, s_1, s_1'} \eta_{2, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
+ \left( k_{s_1} - 1 \right) \left[ 1 - \gamma_{4, s_1, s_1'} - \gamma_{3, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
- \left( k_{s_1} - 1 \right) \left[ \left( 1 - \alpha_{1, s_1, s_1'} \right) \eta_{2, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right] \\
- \left( k_{s_1} - 1 \right) \left[ \alpha_{1, s_1, s_1'} \eta_{2, s_1, s_1'} \left( \eta_{4, s_1, s_1'} - \gamma_{4, s_1, s_1'} \right) \right]
\]

(55)

Especially, when \( i = 1 \), the sensitivity analysis of \( P(g_1) \) with respect to \( \alpha_1 \) and \( \beta_1 \) is needed. According to [41], we can obtain

\[
P(g_1) = \frac{\beta_1}{\alpha_1 + \beta_1}.
\]

Assume the changed parameters are \( \alpha_1' \), \( \beta_1' \) and \( P_{\alpha_1}(g_1), P_{\beta_1}(g_1) \). Then the sensitivity of \( P(g_1) \) with respect to \( \alpha_1 \) and \( \beta_1 \) would be

\[
S_{\alpha_1} = \frac{|P_{\alpha_1}(g_1) - P(g_1)| / P(g_1)}{|\alpha_1' - \alpha_1| / \alpha_1} \quad \text{and} \quad S_{\beta_1} = \frac{|P_{\beta_1}(g_1) - P(g_1)| / P(g_1)}{|\beta_1' - \beta_1| / \beta_1}.
\]

The parameter with \( \max(S_{\gamma_1}, S_{\eta_1}, S_{\eta_2}, S_{\eta_3}) \) when \( i \geq 2 \) or \( \max(S_{\alpha_1}, S_{\beta_1}) \) when \( i = 1 \) is the most sensitive and has the largest impact on \( P(g_1) \).

**B. Bottleneck Analysis**

For M^3Ss, the final quality is the function of internal and external failure probabilities and repair probabilities. Because of product variety, a certain set of failure or repair probabilities contains both internal and external probabilities. For example, for quality failure probabilities with respect to good incoming parts \( \gamma_i \), there are internal probabilities \( \gamma_{i, s_{m-1}} \) \( m = 2, 3, \ldots, n \) and external probabilities \( \gamma_{i, s_{m-1}} \) \( m = 2, 3, \ldots, n \). Then there are several kinds of quality bottlenecks, namely, internal and external quality bottleneck for \( \gamma \) (IQB - \( \gamma \) and EQB - \( \gamma \)), internal and external quality bottleneck for \( \mu \) (IQB - \( \mu \) and EQB - \( \mu \)), internal and external quality bottleneck for \( \eta \) (IQB - \( \eta \) and EQB - \( \eta \)), and internal and external quality bottleneck for \( \theta \) (IQB - \( \theta \) and EQB - \( \theta \)). They are defined as follows.

**Definition 1:** For \( i \neq j \), if \( |\partial P(g_0) / \partial \gamma_{i, s_{m-1}}| > |\partial P(g_0) / \partial \gamma_{j, s_{m-1}}| \), then under sequence \( S_{\gamma}^{i}, \gamma_{i, s_{m-1}} \) \( m = 2, 3, \ldots, n \), \( \gamma \) is the IQB - \( \gamma \) for product \( S_{\gamma}^{m} \), which means when producing product \( S_{\gamma}^{m} \), stage \( i \) is the quality bottleneck with respect to failure probabilities for good incoming parts.

**Definition 2:** For \( i \neq j \), if \( |\partial P(g_0) / \partial \mu_{i, s_{m-1}}| > |\partial P(g_0) / \partial \mu_{j, s_{m-1}}| \), then under sequence \( S_{\mu}^{i}, \mu_{i, s_{m-1}} \) \( m = 2, 3, \ldots, n \), \( \mu \) is the IQB - \( \mu \) for product \( S_{\mu}^{m} \), which means when producing product \( S_{\mu}^{m} \), stage \( i \) is the quality bottleneck with respect to repair probabilities with defective incoming parts.

**Definition 3:** For \( i \neq j \), if \( |\partial P(g_0) / \partial \eta_{i, s_{m-1}}| > |\partial P(g_0) / \partial \eta_{j, s_{m-1}}| \), then under sequence \( S_{\eta}^{i}, \eta_{i, s_{m-1}} \) \( m = 2, 3, \ldots, n \), \( \eta \) is the IQB - \( \eta \) for product \( S_{\eta}^{m} \), which means when producing product \( S_{\eta}^{m} \), stage \( i \) is the quality bottleneck with respect to failure probabilities with defective incoming parts.

**Definition 4:** For \( i \neq j \), if \( |\partial P(g_0) / \partial \theta_{i, s_{m-1}}| > |\partial P(g_0) / \partial \theta_{j, s_{m-1}}| \), then under sequence \( S_{\theta}^{i}, \theta_{i, s_{m-1}} \) \( m = 2, 3, \ldots, n \), \( \theta \) is the IQB - \( \theta \) for product \( S_{\theta}^{m} \), which means when producing
product $S^i_m$, stage $i$ is the quality bottleneck with respect to repair probabilities with defective incoming parts.

**Definition 5:** For $i \neq j$, if $|\partial P(g_4)/\partial \gamma_{1,i-1,S^j_{m-1},S^i_{m}}| > |\partial P(g_4)/\partial \gamma_{1,i-1,S^j_{m-1},S^i_{m}}|$, then under sequence $S^j_{s_{m-1}S^j_{m}}$ is the EQB – $\gamma$ for transition $m$, which means when transitioning from product $S^i_{m-1}$ to $S^i_{m}$, stage $i$ is the quality bottleneck with respect to failure probabilities with good incoming parts.

**Definition 6:** For $i \neq j$, if $|\partial P(g_4)/\partial \mu_{1,i-1,S^j_{m-1},S^i_{m}}| > |\partial P(g_4)/\partial \mu_{1,i-1,S^j_{m-1},S^i_{m}}|$, then under sequence $S^j_{s_{m-1}S^j_{m}}$ is the EQB – $\mu$ for transition $m$, which means when transitioning from product $S^i_{m-1}$ to $S^i_{m}$, stage $i$ is the quality bottleneck with respect to repair probabilities with good incoming parts.

**Definition 7:** For $i \neq j$, if $|\partial P(g_4)/\partial \eta_{1,i-1,S^j_{m-1},S^i_{m}}| > |\partial P(g_4)/\partial \eta_{1,i-1,S^j_{m-1},S^i_{m}}|$, then under sequence $S^j_{s_{m-1}S^j_{m}}$ is the EQB – $\eta$ for transition $m$, which means when transitioning from product $S^i_{m-1}$ to $S^i_{m}$, stage $i$ is the quality bottleneck with respect to failure probabilities with defective incoming parts.

**Definition 8:** For $i \neq j$, if $|\partial P(g_4)/\partial \theta_{1,i-1,S^j_{m-1},S^i_{m}}| > |\partial P(g_4)/\partial \theta_{1,i-1,S^j_{m-1},S^i_{m}}|$, then under sequence $S^j_{s_{m-1}S^j_{m}}$ is EQB – $\theta$ for transition $m$, which means when transitioning from product $S^i_{m-1}$ to $S^i_{m}$, stage $i$ is the quality bottleneck with respect to repair probabilities with defective incoming parts.

For instance, according to (45), the final quality $P(g_4)$ for a two-product-four-stage system is (55), as shown at the top of the previous page.

From (55), we have (56)–(67), as shown at the top of this page.

By comparing (56), (58), and (60), we find the IQB – $\gamma$ for product $S^i_4$ which is the stage
with \( \max[\partial P(g_4)/\partial \gamma_{4,i_1,i_2}] \), \( \partial P(g_4)/\partial \gamma_{3,i_1,i_2} \), \( \partial P(g_4)/\partial \gamma_{2,i_1,i_2} \)\]. Comparing (58), (60), and (62), we obtain the IQB – \( \gamma \) for product \( S_{i_2} \) which is the stage with \( \max\{\partial P(g_4)/\partial \gamma_{4,i_1,i_2}, \partial P(g_4)/\partial \gamma_{3,i_1,i_2}, \partial P(g_4)/\partial \gamma_{2,i_1,i_2}\} \]. We define (56)–(61) as IQB indicators for quality failure probabilities for good incoming parts. Similarly, (62), (64), and (66) indicate the EQB – \( \gamma \) for product \( S_{i_1} \) and (63), (65), and (67) indicate the EQB – \( \gamma \) for product \( S_{i_2} \). We define (62)–(67) as EQB indicators for quality failure probabilities for defective incoming parts.

Two conclusions can be made from (55)–(67).

Conclusion 6: IQB indicators are related to the transition probabilities of both upstream and downstream stages as well as the batch size of the product.

Conclusion 7: EQB indicators are only related to the transition probabilities of both upstream and downstream stages.

VII. CASE STUDY

To validate the applicability of the proposed method, a case study has been conducted. To ensure the confidentiality of the data, all the parameters introduced below have been modified and only used for illustration.

A. Manufacturing System Description

The model is applied to a four-product-five-stage manufacturing system to evaluate the quality performance of the system. The four types of products (valve shells) that need to be manufactured in this system are shown in Fig. 5. There are five dependent stages (OP 10, 20, 30, 40, and 50) that the four types of products will go through. The relationships among these five stages and the quality propagation are discussed in detail in [32].

B. Results and Discussions

1) Model Prediction Error: Taking product 1 for an example. According to historical data, the internal transition probabilities of product 1 are calculated as \( \alpha_1 = 0.05, \beta_1 = 0.94, \gamma_1 = [0.05, 0.1, 0.07, 0.04], \mu_1 = [0.92, 0.87, 0.91, 0.95], \eta_1 = [0.52, 0.55, 0.43, 0.57], \) and \( \theta_1 = [0.45, 0.45, 0.55, 0.45]. \) Based on the developed model, the quality changes of product 1 along the five-stage system can be estimated. The result is shown in Fig. 6.

The probability of producing a good product estimated from the model is 89.46%. The actual final quality based on historical data is 89.71%, and the prediction error is 0.25%. The result demonstrates the effectiveness and practicability of the proposed model. For all of these four products, the average prediction error is about 0.24%.

2) Production Sequence Analysis: According to the historical data, the failure probabilities and repair probabilities approximately follow Bernoulli distribution, which allows us to study the system under Bernoulli case. According to Conclusion 3, to obtain the best sequence, we need to know the various external transition probabilities shown in Table I. They are obtained through the following process.

For a certain part \( j \) processed by \( M_i \), either good or defective, there exist four possible statuses.

1) Both parts \( (j - 1) \) and \( j \) are good; thus, \( M_i \) maintains a good state \( g_i \).
2) Part \( (j - 1) \) is good but part \( j \) is defective; thus, \( M_i \) transits from \( g_i \) to \( d_i \).
3) The part \( (j - 1) \) is defective but part \( j \) is good; thus, \( M_i \) transits from \( d_i \) to \( g_i \).
4) Both parts \( (j - 1) \) and \( j \) are defective; thus, \( M_i \) maintains a defective state \( d_i \).

The transition probabilities are estimated from historical data by calculating the proportions of the status: proportion of status 2) represents \( \alpha, \gamma \), and \( \eta \) while proportion of status 3) refers to \( \beta, \mu \), and \( \theta \). If part \( (j - 1) \) and part \( j \) belong to the same type of product, then the estimated probabilities represent internal transition probabilities. Otherwise, if part \( (j - 1) \) and part \( j \) belong to different types of products, then the estimated probabilities represent external transition probabilities.

According to (38), the sequence with the largest value for four-product-five-stage systems is the best sequence.
The optimization model can be expressed as

\[
\max \sum_{i=1}^{4} \sum_{j=1}^{4} y_{ij} \left[ (1 - \gamma_{5,i,j} - \gamma_{4,i,j})(\eta_{5,i,j} - \gamma_{5,i,j}) - \gamma_{3,i,j}(\eta_{4,i,j} - \gamma_{4,i,j})(\eta_{5,i,j} - \gamma_{5,i,j}) + (1 - \alpha_{1,i,j})(\eta_{4,i,j} - \gamma_{4,i,j})(\eta_{3,i,j} - \gamma_{3,i,j}) \right] \\
\text{s.t. } \sum_{i=1}^{4} y_{ij} = 1, (i \neq j) \\
\sum_{j=1}^{4} y_{ij} = 1, (i \neq j)
\]

where all \( y_{ij} \) = 0 or 1 and all \( \varphi_i > 0 \); when \( y_{ij} = 1 \) product \( i \) is processed right before product \( j \) in the batch cycle, otherwise, \( y_{ij} = 0, \varphi_i > 0 \). By solving the model, we obtain \( y_{1,4} = 1, y_{2,1} = 1, y_{3,2} = 1, y_{4,3} = 1 \), and other \( y_{ij} = 0 \). This result indicates that product 1 is processed before product 4, and product 2 is processed before product 1 while product 3 is processed before product 2, and product 4 is processed before product 3. Then in this case, the best sequence for the four-product-two-stage system would be \( 1 \rightarrow 4 \rightarrow 3 \rightarrow 2 \). Under this sequence, the corresponding probabilities are \( \alpha_{1,1,4}, \alpha_{1,4,3}, \alpha_{1,3,2}, \alpha_{1,2,1}, \gamma_{1,1,4}, \gamma_{4,1,4}, \gamma_{1,3,2}, \gamma_{1,2,1}, \eta_{1,1,4}, \eta_{4,1,4}, \eta_{1,3,2}, \) and \( \eta_{1,2,1} (i = 2, 3, 4, 5) \).

Based on the above analysis, we can obtain the EQB for each kind of parameters. Similar to (62), (64), and (66), we can obtain the derivatives of \( P(g_{5}) \) with respect to each external transition probabilities. As the comparison of derivatives is independent of the batch size, thus, it can be neglected. Therefore as for \( \gamma_{1,1,4} \) the quality failure probability with good incoming parts when transiting from products 1–4, the related derivatives under sequence 1 → 4 → 3 → 2 are calculated as

\[
\frac{\partial P(g_{5})}{\partial \gamma_{1,1,4}} = 0.8034, \frac{\partial P(g_{5})}{\partial \gamma_{4,1,4}} = 0.7695, \frac{\partial P(g_{5})}{\partial \gamma_{5,1,4}} = 0.2856, \text{ and } \frac{\partial P(g_{5})}{\partial \gamma_{2,1,4}} = 0.1671.
\]

Since \( \frac{\partial P(g_{5})}{\partial \gamma_{1,1,4}} \) has the largest value, the fifth stage OP50 is the EQB for the quality failure probability with good incoming parts when transiting from products 1–4 in this case. The improvement of \( \gamma_{1,1,4} \) can bring the largest benefit to the final quality.

3) Bottleneck Analysis: Also, take product 1 for an example. The quality has the biggest decrease in the third stage (Fig. 6). Taking the third stage into consideration, we can do the sensitivity analysis to find out which transition probability \( P(g_{5}) \) is most sensitive. In this case, \( \gamma_{3}, \mu_{3}, \eta_{3}, \) and \( \theta_{3} \) are increased or decreased by given percentages and the sensitivities with respect to the four parameters at 10% are \( S_{\gamma_{3}} = 9.43\%, S_{\mu_{3}} = 12.13\%, S_{\eta_{3}} = 4.21\%, \) and \( S_{\theta_{3}} = 0.61\% \), respectively, which indicates that the quality \( P(g_{5}) \) is most sensitive to the quality failure probability for good incoming parts \( \mu_{3} \). Based on the model and Definitions 1–4, we can find the IQB for this system. For the repair probability with good incoming parts \( \mu_{i} \), the relationships between \( \mu_{i} \) and \( P(g_{4}) \) are shown in Fig. 7.

### Table I

<table>
<thead>
<tr>
<th></th>
<th>(1,2)</th>
<th>(1,3)</th>
<th>(1,4)</th>
<th>(2,1)</th>
<th>(2,3)</th>
<th>(2,4)</th>
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<td>0.0667</td>
<td>0.0792</td>
<td>0.0989</td>
<td>0.0717</td>
<td>0.0598</td>
<td>0.0952</td>
</tr>
<tr>
<td>( a_{2} )</td>
<td>0.0643</td>
<td>0.0852</td>
<td>0.0972</td>
<td>0.0916</td>
<td>0.0837</td>
<td>0.0779</td>
</tr>
<tr>
<td>( a_{3} )</td>
<td>0.1259</td>
<td>0.1208</td>
<td>0.1390</td>
<td>0.1010</td>
<td>0.1409</td>
<td>0.1230</td>
</tr>
<tr>
<td>( a_{4} )</td>
<td>0.1068</td>
<td>0.1120</td>
<td>0.0840</td>
<td>0.0958</td>
<td>0.0888</td>
<td>0.1083</td>
</tr>
<tr>
<td>( a_{5} )</td>
<td>0.0478</td>
<td>0.0829</td>
<td>0.0623</td>
<td>0.0458</td>
<td>0.0589</td>
<td>0.0635</td>
</tr>
<tr>
<td>( \eta_{1} )</td>
<td>0.6085</td>
<td>0.5775</td>
<td>0.5711</td>
<td>0.5482</td>
<td>0.6118</td>
<td>0.6887</td>
</tr>
<tr>
<td>( \eta_{2} )</td>
<td>0.6860</td>
<td>0.6491</td>
<td>0.6948</td>
<td>0.6111</td>
<td>0.6087</td>
<td>0.7403</td>
</tr>
<tr>
<td>( \eta_{3} )</td>
<td>0.5710</td>
<td>0.5818</td>
<td>0.5981</td>
<td>0.4468</td>
<td>0.5520</td>
<td>0.6047</td>
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<tr>
<td>( \eta_{4} )</td>
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<td>0.6849</td>
<td>0.7113</td>
<td>0.7278</td>
<td>0.6925</td>
<td>0.6930</td>
</tr>
</tbody>
</table>

### References

1. IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS: SYSTEMS
From Fig. 7, we can see that changes of $\mu_5$ have the greatest impact on the final quality $P(g_5)$. The derivatives of $P(g_5)$ also show the same result. The corresponding derivatives are $[\partial P(g_5)/\partial \mu_2] = 0.0065$, $[\partial P(g_5)/\partial \mu_3] = 0.0255$, $[\partial P(g_5)/\partial \mu_4] = 0.0570$, and $[\partial P(g_5)/\partial \mu_5] = 0.0938$. As $[\partial P(g_5)/\partial \mu_5]$ is the largest among these derivatives, the fifth stage OP50 is the IQB for repair probability with good incoming parts $\mu_i$ in this case. This means changes of $\mu_5$ can lead to the largest improvement to the final quality.

VIII. Conclusion

$M^3$Ss have been widely applied in industry. It is very important to develop a proper method to evaluate the quality performance of $M^3$Ss. This paper is devoted to modeling and analyzing steady-state $M^3$Ss for quality improvement and filling the gap between FMSs and quality propagation among multiple stages. Two novel Markov models for multiproduct-two-stage systems and multiproduct-multistage systems are developed for quality improvement. The models take both product flexibility and quality propagation into consideration. Several important quality properties including production strategy, product sequence, and quality bottleneck identification are analyzed and a few practical conclusions are noted to provide some insights on quality improvement. Finally, a case study on valve shells is conducted to illustrated the effectiveness of the proposed models. The average prediction error of the models is about 0.24%.

Based on the proposed models in this paper, future work can be conducted as follows.

1) The Markov models can be extended to the one charactering the transient-state quality performance. A Markov model is desirable to be developed for both steady-state and transient-state quality performance.

2) Some other properties in $M^3$Ss for quality improvement can be investigated, such as monotonicity, settling time, and quality loss.

3) The future work also can be directed to modeling and analysis of serial-parallel $M^3$Ss.

APPENDIX

Proofs

A. Proof of Conclusion 3

According to (39), when there are $n$ types of products, under BP, the difference between the probabilities of producing good products with sequence $S'$ and $S^m$ is (A1), as shown at the top of the next page, where

$$\sum_{i=1}^{n} k_{ij}' = \sum_{i=1}^{n} k_{ij}^m \quad \text{(A2)}$$

$$\sum_{i=1}^{n} \left( k_{ij}' - 1 \right) \left( 1 - \alpha_{1j}' s_i' \gamma_{1j}' s_i' + \alpha_{ij}' s_i' \gamma_{ij}' s_i' - \gamma_{ij}' s_i' \right)$$

$$= \sum_{i=1}^{n} \left( k_{ij}^m - 1 \right) \left( 1 - \alpha_{1j}^m s_i^m \gamma_{1j}^m s_i^m + \alpha_{ij}^m s_i^m \gamma_{ij}^m s_i^m - \gamma_{ij}^m s_i^m \right). \quad \text{(A3)}$$

Thus, A(1) can be simplified as (A4), as shown at the top of the next page.

On the other hand, under SP, the difference between probabilities of producing good product with sequence $S'$ and $S^m$ is (A5), as shown at the top of the next page.

Comparing A(1) with A(5), it can be seen that they have the same signature, which means the best sequences under BP and SP are consistent with each other: $P(g_{bop}') - P(g_{bop}^m) \iff P(g_{sop}^1) - P(g_{sop}^m).$

B. Proof of Conclusion 4

Under Bernoulli case, assume that $\delta_{ij} = 1 - \alpha_{1j}' s_i' - \beta_{1j}' s_i'$, $\delta_{ij}^2 = 1 - \gamma_{1j}' s_i' - \mu_{2j}' s_i'$, $\delta_{ij}^3 = 1 - \eta_{2j}' s_i' - \theta_{2j}' s_i'$ and the...
\[
P(s_{bp}^I) - P(s_{bp}^m) = \sum_{i=1}^n (k_{s_{bp}^i} - 1) \left( 1 - \alpha_{1,s_{bp}^i} s_{bp}^i s_{bp}^i + \alpha_{1,s_{bp}^i} s_{bp}^i s_{bp}^i \right) \\
+ \sum_{i=1}^n \left( 1 - \alpha_{1,s_{bp}^i} s_{bp}^i s_{bp}^i + \alpha_{1,s_{bp}^i} s_{bp}^i s_{bp}^i \right) \\
- \sum_{i=1}^n \left( 1 - \alpha_{1,s_{bp}^i} s_{bp}^i s_{bp}^i + \alpha_{1,s_{bp}^i} s_{bp}^i s_{bp}^i \right)
\]

\[
P(s_{ss}^I) - P(s_{ss}^m) = \sum_{i=1}^n \left( 1 - \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i + \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i \right) \\
+ \sum_{i=1}^n \left( 1 - \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i + \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i \right) \\
- \sum_{i=1}^n \left( 1 - \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i + \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i \right)
\]

\[
P(s_{ss}^I) - P(s_{ss}^m) = \sum_{i=1}^n \left( 1 - \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i + \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i \right) \\
+ \sum_{i=1}^n \left( 1 - \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i + \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i \right) \\
- \sum_{i=1}^n \left( 1 - \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i + \alpha_{1,s_{ss}^i} s_{ss}^i s_{ss}^i \right)
\]

\[
C = \begin{bmatrix}
0 & C_{1,2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & C_{i,k_{s_{bp}^i}} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & C_{i+1,2} & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
C_{1,1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

(A6)

where (A7), as shown at the top of the next page.

\[C_{i,j}\] can be written as the summation of \[C_{i,j}^1\] and \[C_{i,j}^2\], and (A8) and (A9), as shown at the top of the next page.

According to (26)

\[
B = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}
\]

(A10)

\[
B_1 = B_2 = \cdots = B_n = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{K} \end{bmatrix}
\]

(A11)

Then for a general case (A12), as shown at the top of the next page.

In the Bernoulli case where \(C_{i,j}^2 = 0\), (A13), as shown at the top of the page following the next page.

Defining function \(F\) as the summation of the first and second column of the fourth row in a 4 \times 4 matrix, we have

\[
P(g') = \frac{1}{K} \sum_{i=1}^n k_{i} F \left[ \left( C_{i,j}^1 + C_{i,j}^2 \right)^{-1} \right]
\]

(A14)

\[
\chi' = \frac{1}{K} \sum_{i=1}^n k_{i} F \left[ \left( C_{i,j}^1 \right)^{-1} \right]
\]

(A15)

\[
P(g') - \frac{1}{1 + \delta_{\max}} \chi' = \frac{1}{K(1 + \delta_{\max})} \chi' \]

(A16)

Calculate the block matrixes \((C_{i,j}^1 + C_{i,j}^2)^{-1}\) and \((C_{i,j}^1)^{-1}\), we obtain that when \(\delta_{\max} \leq 11(1 + \delta_{\max}) F(C_{i,j}^1 + C_{i,j}^2)^{-1} - F(C_{i,j}^1)^{-1} \geq 0\) which means \(P(g') - \left[ 1/(1 + \delta_{\max}) \right] \chi' \geq 0\).

Similarly, \(P(g') - \left[ 1/(1 + \delta_{\max}) \right] \chi' \leq 0\) can be proved.
\[ C_{ij} = \begin{bmatrix}
1 - \alpha_{i,j} & (1 - \gamma_{j}) & (1 - \gamma_{j}) & 1 & 1 \\
\beta_{i,j} & 1 - \eta_{i,j} & 1 - \eta_{i,j} & \alpha_{i,j} & \beta_{i,j} \\
\beta_{i,j} & 1 - \eta_{i,j} & 1 - \eta_{i,j} & \alpha_{i,j} & \beta_{i,j} \\
1 - \alpha_{i,j} & (1 - \gamma_{j}) & (1 - \gamma_{j}) & 1 & 1 \\
\beta_{i,j} & 1 - \eta_{i,j} & 1 - \eta_{i,j} & \alpha_{i,j} & \beta_{i,j}
\end{bmatrix}
\]

\[ C_{ij}^1 = \begin{bmatrix}
1 - \alpha_{i,j} & (1 - \gamma_{j}) & (1 - \gamma_{j}) & 1 & 1 \\
\beta_{i,j} & 1 - \eta_{i,j} & 1 - \eta_{i,j} & \alpha_{i,j} & \beta_{i,j} \\
\beta_{i,j} & 1 - \eta_{i,j} & 1 - \eta_{i,j} & \alpha_{i,j} & \beta_{i,j} \\
1 - \alpha_{i,j} & (1 - \gamma_{j}) & (1 - \gamma_{j}) & 1 & 1 \\
\beta_{i,j} & 1 - \eta_{i,j} & 1 - \eta_{i,j} & \alpha_{i,j} & \beta_{i,j}
\end{bmatrix}
\]

\[ C_{ij}^2 = \begin{bmatrix}
1 - \alpha_{i,j} & (1 - \gamma_{j}) & (1 - \gamma_{j}) & 1 & 1 \\
\beta_{i,j} & 1 - \eta_{i,j} & 1 - \eta_{i,j} & \alpha_{i,j} & \beta_{i,j} \\
\beta_{i,j} & 1 - \eta_{i,j} & 1 - \eta_{i,j} & \alpha_{i,j} & \beta_{i,j} \\
1 - \alpha_{i,j} & (1 - \gamma_{j}) & (1 - \gamma_{j}) & 1 & 1 \\
\beta_{i,j} & 1 - \eta_{i,j} & 1 - \eta_{i,j} & \alpha_{i,j} & \beta_{i,j}
\end{bmatrix}
\]

\[ X = BC^{-1} = B (C^1 + C^2)^{-1} = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}
\]

\[ \times \begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & (C_{1,1} + C_{1,1})^{-1} \\
(C_{1,2} + C_{1,2})^{-1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & (C_{i,1} + C_{i,1})^{-1} & \cdots & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & (C_{i+1,1} + C_{i+1,1})^{-1} & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & (C_{i+1,2} + C_{i+1,2})^{-1} & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

\[ (A7) \]

\[ (A8) \]

\[ (A9) \]

**C. Proof of Conclusion 5**

According to (39) and (40), \( \chi_{bp}^l \), \( \chi_{bp}^m \), \( \chi_{sp}^l \), and \( \chi_{sp}^m \) can be obtained, (A17)–(A20), as shown at the top of the next page.

Then according to (A2) and (A3)

\[ \frac{\chi_{bs}^l - \chi_{bs}^m}{\chi_{bs}^l + \chi_{bs}^m} = \frac{\chi_{bp}^l - \chi_{bp}^m}{\chi_{bp}^l + \chi_{bp}^m} \]
\[ X' = BC^{-1} = B \left( C^1 \right)^{-1} \]

\[
\mathbf{X} = \begin{bmatrix}
B_1 & B_2 & \cdots & B_n
\end{bmatrix}
\]

\[
\chi_{bp}^l = P\left( g_{bp}^l \right) = \frac{\sum_{i=1}^{n} (k_{S_j} - 1) \left( 1 - \alpha_{1,S_j} \gamma_{1,S_j,1} + \alpha_{1,S_j} \gamma_{1,S_j,1} - \gamma_{2,S_j,1} \right)}{\sum_{i=1}^{n} k_{S_j}}
\]

\[
\chi_{sp}^l = P\left( g_{sp}^l \right) = \frac{\sum_{i=1}^{n} (k_{S_j} - 1) \left( 1 - \alpha_{1,S_j} \gamma_{1,S_j,1} + \alpha_{1,S_j} \gamma_{1,S_j,1} - \gamma_{2,S_j,1} \right)}{\sum_{i=1}^{n} k_{S_j}}
\]

\[
\chi_{bp}^m = P\left( g_{bp}^m \right) = \frac{\sum_{i=1}^{n} (k_{S_j} - 1) \left( 1 - \alpha_{1,S_j} \gamma_{1,S_j,1} + \alpha_{1,S_j} \gamma_{1,S_j,1} - \gamma_{2,S_j,1} \right)}{\sum_{i=1}^{n} k_{S_j}}
\]

\[
\chi_{sp}^m = P\left( g_{sp}^m \right) = \frac{\sum_{i=1}^{n} (k_{S_j} - 1) \left( 1 - \alpha_{1,S_j} \gamma_{1,S_j,1} + \alpha_{1,S_j} \gamma_{1,S_j,1} - \gamma_{2,S_j,1} \right)}{\sum_{i=1}^{n} k_{S_j}}
\]

Based on Conclusion 4

\[
\frac{\chi_{bp}^l - \chi_{bp}^m}{1 - \delta_{\text{max}}} \leq \frac{\chi_{bp}^l + \chi_{bp}^m}{\chi_{bp}^l - \chi_{bp}^m} < \frac{\chi_{bp}^l - \chi_{bp}^m}{1 - \delta_{\text{max}}}< P\left( g_{bp}^l \right) - P\left( g_{bp}^m \right) < \frac{\chi_{bp}^l - \chi_{bp}^m}{1 - \delta_{\text{max}}}
\]

when \( 0 < \delta_{\text{max}} < \frac{\chi_{bp}^l - \chi_{bp}^m}{\chi_{ss}^l + \chi_{ss}^m} \), we have \( \chi_{ss}^l - \chi_{ss}^m > 0 \) and \( \chi_{bp}^l - \chi_{bp}^m > 0 \), then
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Shichang Du (M’16) received the B.S. and M.S.E. degrees in mechanical engineering from the Hefei University of Technology, Hefei, China, in 2000 and 2003, respectively, and the Ph.D. degree in industrial engineering and management from Shanghai Jiao Tong University, Shanghai, China, in 2008. He was a Visiting Scholar with the University of Michigan, Ann Arbor, MI, USA, from 2006 to 2007. He is an Associate Professor with Shanghai Jiao Tong University. His current research interests include quality and reliability engineering, quality control with analysis of error flow, and monitoring and diagnosis of manufacturing process.

Rui Xu received the B.S. degree from the School of Mechanical Engineering, Shanghai Jiao Tong University, Shanghai, China, in 2013, where she is currently pursuing master’s degree in industrial engineering and management. Her current research interests include Markov modeling and analysis of multistage manufacturing systems.

Lin Li (M’12) received the B.S. degree in mechanical engineering from Shanghai Jiao Tong University, Shanghai, China, in 2001, and the M.S.E. degrees in mechanical engineering and industrial and operations engineering and the Ph.D. degree in mechanical engineering from the University of Michigan, Ann Arbor, MI, USA, in 2003, 2005, and 2007, respectively. He is currently an Associate Professor with the Department of Mechanical and Industrial Engineering and the Director of Sustainable Manufacturing Systems Research Laboratory, University of Illinois at Chicago, Chicago, IL, USA. His current research interests include sustainable manufacturing systems, reliability engineering, intelligent maintenance systems, and healthcare systems.